

A CHANGE OF MEASURE PRESERVING THE AFFINE STRUCTURE IN THE BNS MODEL FOR COMMODITY MARKETS

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ABSTRACT. For a commodity spot price dynamics given by an Ornstein-Uhlenbeck process with Barndorff-Nielsen and Shephard stochastic volatility, we price forwards using a class of pricing measures that simultaneously allow for change of level and speed in the mean reversion of both the price and the volatility. The risk premium is derived in the case of arithmetic and geometric spot price processes, and it is demonstrated that we can provide flexible shapes that is typically observed in energy markets. In particular, our pricing measure preserves the affine model structure and decomposes into a price and volatility risk premium, and in the geometric spot price model we need to resort to a detailed analysis of a system of Riccati equations, for which we show existence and uniqueness of solution and asymptotic properties that explains the possible risk premium profiles. Among the typical shapes, the risk premium allows for a stochastic change of sign, and can attain positive values in the short end of the forward market and negative in the long end.

1. INTRODUCTION

Benth and Ortiz-Latorre [9] analysed a structure preserving class of pricing measures for Ornstein-Uhlenbeck (OU) processes with applications to forward pricing in commodity markets. In particular, they considered multi-factor OU models driven by Lévy processes having positive jumps (so-called subordinators) or Brownian motions for the spot price dynamics and analysed the risk premium when the level and speed of mean reversion in these factor processes were changed.

In this paper we continue this study for OU processes driven by Brownian motion, but with a stochastic volatility perturbing the driving noise. The stochastic volatility process is modelled again as an OU process, but driven by a subordinator. This class of stochastic volatility models were first introduced by Barndorff-Nielsen and Shephard [1] for equity prices, and later analysed by Benth [2] in commodity markets. Indeed, the present paper is considering a class of pricing measures preserving the affine structure of the spot price model analysed in Benth [2].

Our spot price dynamics is a generalization of the Schwartz model (see Schwartz [23]) to account for stochastic volatility. The Schwartz model have been applied to many different commodity markets, including oil (see Schwartz [23]), power (see Lucia and Schwartz [20]), weather (see Benth and Šaltytė Benth [4]) and freight (see Benth, Koekebakker and Taib [8]). Like Lucia and Schwartz [20], we analyse both geometric and arithmetic models for the spot price evolution. There exists many extensions of the model, typically allowing for more factors in the spot price dynamics, as well as modelling the convenience yield and interest rates (see Eydeland and Wolyniec [11] and Geman [13] for more on such models). In Benth [2], the Schwartz model with stochastic volatility has been applied to model empirically UK gas prices. Also other stochastic volatility models like the Heston have been suggested in the context of commodity markets (see Eydeland and Wolyniec [11] and Geman [13] for a discussion and further references).

The class of pricing measures we study here allows for a simultaneous change of speed and level of mean reversion for both the (logarithmic) spot price and the stochastic volatility process. The mean reversion level can be flexibly shifted up or down, while the speed of mean reversion can be slowed down. It significantly extends the Esscher transform, which only allows for changes in the level of mean reversion. Indeed, it decomposes the risk premium into a price and volatility premium. It has been studied

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empirically in some commodity markets for multi-factor models in Benth, Cartea and Pedraz [7]. As we show, the class of pricing measures preserves the affine structure of the model, but leads to a rather complex stochastic driver for the stochastic volatility. For the arithmetic spot model we can derive analytic forward prices and risk premium curves. On the other hand, the geometric model is far more complex, but the affine structure can be exploited to reduce the forward pricing to solving a system of Riccati equations by resorting to the theory of Kallsen and Muhle-Karbe [18]. The forward price becomes a function of both the spot and the volatility, and has a deterministic asymptotic dynamics when we are far from maturity.

By careful analysis of the associated system of Riccati equations, we can study the implied risk premium of our class of measure change as a function of its parameters. The risk premium is defined as the difference between the forward price and the predicted spot price at maturity, and is a notion of great importance in commodity markets since it measures the price for entering a forward hedge position in the commodity (see e.g. Geman [13] for more on this). In particular, under rather mild assumptions on the parameters, we can show that the risk premium may change sign stochastically, and may be positive for short times to maturity and negative when maturity is farther out in time. This is a profile of the risk premium that one may expect in power markets based on both economical and empirical findings. Geman and Vasicek [14] argue that retailers in the power market may induce a hedging pressure by entering long positions in forwards to protect themselves against sudden price increases (spikes). This may lead to positive risk premia, whereas producers induce a negative premium in the long end of the forward curve since they hedge by selling their production. This economic argument for a positive premium in the short end is backed up by empirical evidence from the German power market found in Benth, Cartea and Kiesel [6]. In the geometric model, we show that the sign of the risk premium depends explicitly on the current level of the logarithmic spot price.

We recover the Esscher transform in a special case of our pricing measure. The Esscher transform is a popular tool for introducing a pricing measure in commodity markets, or, equivalently, to model the risk premium. For constant market prices of risk, which are defined as the shift in level of mean reversion, we preserve the affine structure of the model as well as the Lévy property of the driving noises of the two OU processes that we consider (indeed, the spot price dynamics is driven by a Brownian motion). We find such pricing measures in for example Lucia and Schwartz [20], Kolos and Ronn [19] and Schwartz and Smith [24]. We refer the reader to Benth, Šaltytė Benth and Koekebakker [5] for a thorough discussion and references to the application of Esscher transform in power and related markets. We note that the Esscher transform was first introduced and applied to insurance as a tool to model the premium charged for covering a given risk exposure and later adopted in pricing in incomplete financial markets (see Gerber and Shiu [15]). In many ways, in markets where the underlying commodity is not storable (that is, cannot be traded in a portfolio), the pricing of forwards and futures can be viewed as an exercise in determining an insurance premium. Our more general change of measure is still structure preserving, however, risk is priced also in the sense that one slows down the speed of mean reversion. Such a reduction allow the random fluctuations of the spot and the stochastic volatility last longer under the pricing measure than under the objective probability, and thus spreads out the risk.

Although our analysis has a clear focus on the stylized facts of the risk premium in power markets, the proposed class of pricing measures is clearly also relevant in other commodity markets. As already mentioned, markets like weather and freight share some similarities with power in that the underlying "spot" is not storable. Also in more classical commodity markets like oil and gas there are evidences of stochastic volatility and spot prices following a mean-reversion dynamics, at least as a component of the spot. Moreover, in the arithmetic case our analysis relates to the concept of unspanned volatility in commodity markets, extensively studied by Trolle and Schwartz [25]. The forward price will not depend on the stochastic volatility factor, and hence one cannot hedge options by forwards alone. Interestingly, the corresponding geometric model will in fact span the stochastic volatility.

We present our results as follows. In the next Section we present the spot model, and follow up in Section 3 with introducing our pricing measure validating that this is indeed an equivalent probability. In Section 4 we derive forward prices under the arithmetic spot price model, and analyse the implied risk premium. Section 5 considers the corresponding forward prices and the implied risk premium for the

geometric spot price model. Here we exploit the affine structure of the model to analyse the associated Riccati equation, and provide insight into the potential risk premium profiles that our set-up can generate. Both Section 4 and 5 have numerous empirical examples.

2. MATHEMATICAL MODEL

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ is a complete filtered probability space, where $T > 0$ is a fixed finite time horizon. On this probability space there are defined W , a standard Wiener process, and L , a pure jump Lévy subordinator with finite expectation, that is a Lévy process with the following Lévy-Itô representation $L(t) = \int_0^t \int_0^\infty z N^L(ds, dz)$, $t \in [0, T]$, where $N^L(ds, dz)$ is a Poisson random measure with Lévy measure ℓ satisfying $\int_0^\infty z \ell(dz) < \infty$. We shall suppose that W and L are independent of each other.

As we are going to consider an Esscher change of measure and geometric spot price models, we introduce the following assumption on the existence of exponential moments of L .

Assumption 1. *Suppose that*

$$\Theta_L \triangleq \sup\{\theta \in \mathbb{R}_+ : \mathbb{E}[e^{\theta L(1)}] < \infty\}, \quad (2.1)$$

is a constant strictly greater than one, which may be ∞ .

Some remarks are in order.

Remark 2.1. In $(-\infty, \Theta_L)$ the cumulant (or log moment generating) function $\kappa_L(\theta) \triangleq \log \mathbb{E}_P[e^{\theta L(1)}]$ is well defined and analytic. As $0 \in (-\infty, \Theta_L)$, L has moments of all orders. Also, $\kappa_L(\theta)$ is convex, which yields that $\kappa_L''(\theta) \geq 0$ and, hence, that $\kappa_L'(\theta)$ is non decreasing. Finally, as a consequence of $L \geq 0$, a.s., we have that $\kappa_L'(\theta)$ is non negative.

Remark 2.2. Thanks to the Lévy-Kintchine representation of L we can express $\kappa_L(\theta)$ and its derivatives in terms of the Lévy measure ℓ . We have that for $\theta \in (-\infty, \Theta_L)$

$$\begin{aligned} \kappa_L(\theta) &= \int_0^\infty (e^{\theta z} - 1) \ell(dz) < \infty, \\ \kappa_L^{(n)}(\theta) &= \int_0^\infty z^n e^{\theta z} \ell(dz) < \infty, \quad n \in \mathbb{N}, \end{aligned}$$

showing, in fact, that $\kappa_L^{(n)}(\theta) > 0$, $n \in \mathbb{N}$.

Consider the OU processes

$$X(t) = X(0) - \alpha \int_0^t X(s) ds + \int_0^t \sigma(s) dW(s) \quad t \in [0, T], \quad (2.2)$$

$$\sigma^2(t) = \sigma^2(0) - \rho \int_0^t \sigma^2(s) ds + L(t), \quad t \in [0, T], \quad (2.3)$$

with $\alpha, \rho > 0$, $X(0) \in \mathbb{R}$, $\sigma^2(0) > 0$. Note that, in equation 2.2, X is written as a sum of a finite variation process and a martingale. We may also rewrite equation 2.3 as a sum of a finite variation part and pure jump martingale

$$\sigma^2(t) = \sigma^2(0) + \int_0^t (\kappa_L'(0) - \rho \sigma^2(s)) ds + \int_0^t \int_0^\infty z \tilde{N}^L(ds, dz), \quad t \in [0, T],$$

where $\tilde{N}^L(ds, dz) \triangleq N^L(ds, dz) - ds \ell(dz)$ is the compensated version of $N^L(ds, dz)$. In the notation of Shiryaev [22], page 669, the predictable characteristic triplets (with respect to the pseudo truncation function $g(x) = x$) of X and σ^2 are given by

$$(B^X(t), C^X(t), \nu^X(dt, dz)) = (-\alpha \int_0^t X(s) ds, \int_0^t \sigma^2(s) ds, 0), \quad t \in [0, T],$$

and

$$(B^{\sigma^2}(t), C^{\sigma^2}(t), \nu^{\sigma^2}(dt, dz)) = (\int_0^t (\kappa'_L(0) - \rho\sigma^2(s))ds, 0, \ell(dz)dt), \quad t \in [0, T],$$

respectively. In addition, applying Itô's Formula to $e^{\alpha t}X(t)$ and $e^{\rho t}\sigma^2(t)$, one can find the following explicit expressions for $X(t)$ and $\sigma^2(t)$

$$X(t) = X(s)e^{-\alpha(t-s)} + \int_s^t \sigma(u)e^{-\alpha(t-u)}dW(u), \quad (2.4)$$

$$\sigma^2(t) = \sigma^2(s)e^{-\rho(t-s)} + \frac{\kappa'_L(0)}{\rho}(1 - e^{-\rho(t-s)}) + \int_s^t \int_0^\infty e^{-\rho(t-u)} z \tilde{N}^L(du, dz), \quad (2.5)$$

where $0 \leq s \leq t \leq T$.

Using the notation in Kallsen and Muhle-Karbe [18], we have that the process $Z = (Z^1(t), Z_2(t)) \triangleq (\sigma^2(t), X(t))$ has affine differential characteristics given by

$$\begin{aligned} \beta_0 &= \begin{pmatrix} \kappa'_L(0) \\ 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_0(A) = \int_0^\infty \mathbf{1}_A(z, 0)\ell(dz), \forall A \in \mathcal{B}(\mathbb{R}^2) \\ \beta_1 &= \begin{pmatrix} -\rho \\ 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_1(A) \equiv 0, \forall A \in \mathcal{B}(\mathbb{R}^2), \\ \beta_2 &= \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_2(A) \equiv 0, \forall A \in \mathcal{B}(\mathbb{R}^2). \end{aligned}$$

These characteristics are admissible and correspond to an affine process in $\mathbb{R}_+ \times \mathbb{R}$.

3. THE CHANGE OF MEASURE

We will consider a parametrized family of measure changes which will allow us to simultaneously modify the speed and the level of mean reversion in equations (2.2) and (2.3). The density processes of these measure changes will be determined by the stochastic exponential of certain martingales. To this end, consider the following family of kernels

$$G_{\theta_1, \beta_1}(t) \triangleq \sigma^{-1}(t) (\theta_1 + \alpha\beta_1 X(t)), \quad t \in [0, T], \quad (3.1)$$

$$H_{\theta_2, \beta_2}(t, z) \triangleq e^{\theta_2 z} \left(1 + \frac{\rho\beta_2}{\kappa''_L(\theta_2)} z\sigma^2(t-) \right), \quad t \in [0, T], z \in \mathbb{R}. \quad (3.2)$$

The parameters $\bar{\beta} \triangleq (\beta_1, \beta_2)$ and $\bar{\theta} \triangleq (\theta_1, \theta_2)$ will take values on the following sets $\bar{\beta} \in [0, 1]^2$, $\bar{\theta} \in \bar{D}_L \triangleq \mathbb{R} \times D_L$, where $D_L \triangleq (-\infty, \Theta_L/2)$ and Θ_L is given by equation (2.1). By Assumption 1 and Remarks 2.1 and 2.2, these kernels are well defined.

Example 3.1. Typical examples of ℓ , Θ_L and D_L are the following:

- (1) Bounded support: L has a jump of size a , i.e. $\ell = \delta_a$. In this case $\Theta_L = \infty$ and $D_L = \mathbb{R}$.
- (2) Finite activity: L is a compound Poisson process with exponential jumps, i.e., $\ell(dz) = ce^{-\lambda z} \mathbf{1}_{(0, \infty)}(dz)$, for some $c > 0$ and $\lambda > 0$. In this case $\Theta_L = \lambda$ and $D_L = (-\infty, \lambda/2)$.
- (3) Infinite activity: L is a tempered stable subordinator, i.e., $\ell(dz) = cz^{-(1+\alpha)}e^{-\lambda z} \mathbf{1}_{(0, \infty)}(dz)$, for some $c > 0$, $\lambda > 0$ and $\alpha \in [0, 1)$. In this case also $\Theta_L = \lambda$ and $D_L = (-\infty, \lambda/2)$.

Next, for $\bar{\beta} \in [0, 1]^2$, $\bar{\theta} \in \bar{D}_L$, define the following family of Wiener and Poisson integrals

$$\begin{aligned} \tilde{G}_{\theta_1, \beta_1}(t) &\triangleq \int_0^t G_{\theta_1, \beta_1}(s)dW(s), \quad t \in [0, T], \\ \tilde{H}_{\theta_2, \beta_2}(t) &\triangleq \int_0^t \int_0^\infty (H_{\theta_2, \beta_2}(s, z) - 1) \tilde{N}^L(ds, dz), \quad t \in [0, T], \end{aligned}$$

associated to the kernels G_{θ_1, β_1} and H_{θ_2, β_2} , respectively.

We propose a family of measure changes given by $Q_{\bar{\theta}, \bar{\beta}} \sim P, \bar{\beta} \in [0, 1]^2, \bar{\theta} \in \bar{D}_L$, with

$$\left. \frac{dQ_{\bar{\theta}, \bar{\beta}}}{dP} \right|_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t), \quad t \in [0, T]. \quad (3.3)$$

Let us assume for a moment that $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})$ is a strictly positive true martingale (this will be proven in Theorem 3.4 below): Then, by Girsanov's theorem for semimartingales (Theorems 1 and 3, pages 702 and 703 in Shiryaev [22]), the process $X(t)$ and $\sigma^2(t)$ become

$$\begin{aligned} X(t) &= X(0) + B_{Q_{\bar{\theta}, \bar{\beta}}}^X(t) + \sigma(t)W_{Q_{\bar{\theta}, \bar{\beta}}}(t), \quad t \in [0, T], \\ \sigma^2(t) &= \sigma^2(0) + B_{Q_{\bar{\theta}, \bar{\beta}}}^{\sigma^2}(t) + \int_0^t \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz), \quad t \in [0, T], \end{aligned}$$

with

$$B_{Q_{\bar{\theta}, \bar{\beta}}}^X(t) = \int_0^t (\theta_1 - \alpha(1 - \beta_1)X(s))ds, \quad t \in [0, T], \quad (3.4)$$

$$\begin{aligned} B_{Q_{\bar{\theta}, \bar{\beta}}}^{\sigma^2}(t) &= \int_0^t (\kappa'_L(0) - \rho\sigma^2(s))ds + \int_0^t \int_0^\infty z(H_{\theta_2, \beta_2}(s, z) - 1)\ell(dz)ds \\ &= \int_0^t \{(\kappa'_L(0) - \rho\sigma^2(s)) + \int_0^\infty z(e^{\theta_2 z} - 1)\ell(dz) \\ &\quad + \frac{\rho\beta_2}{\kappa''_L(\theta_2)} \int_0^\infty z^2 e^{\theta_2 z} \ell(dz)\sigma^2(s-)\}ds \\ &= \int_0^t (\kappa'_L(\theta_2) - \rho(1 - \beta_2)\sigma^2(s))ds, \quad t \in [0, T], \end{aligned} \quad (3.5)$$

where $W_{Q_{\bar{\theta}, \bar{\beta}}}$ is a $Q_{\bar{\theta}, \bar{\beta}}$ -standard Wiener process and the $Q_{\bar{\theta}, \bar{\beta}}$ -compensator measure of σ^2 (and L) is

$$v_{Q_{\bar{\theta}, \bar{\beta}}}^{\sigma^2}(dt, dz) = v_{Q_{\bar{\theta}, \bar{\beta}}}^L(dt, dz) = H_{\theta_2, \beta_2}(t, z)\ell(dz)dt.$$

In conclusion, the semimartingale triplet for X and σ^2 under $Q_{\bar{\theta}, \bar{\beta}}$ are given by $(B_{Q_{\bar{\theta}, \bar{\beta}}}^X, \int_0^\cdot \sigma^2(s)ds, 0)$ and $(B_{Q_{\bar{\theta}, \bar{\beta}}}^{\sigma^2}, 0, v_{Q_{\bar{\theta}, \bar{\beta}}}^{\sigma^2})$, respectively.

Remark 3.2. Under $Q_{\bar{\theta}, \bar{\beta}}$, the process $Z = (\sigma^2(t), X(t))$ is affine with differential characteristics given by

$$\begin{aligned} \beta_0 &= \begin{pmatrix} \kappa'_L(\theta_2) \\ \theta_1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_0(A) = \int_0^\infty \mathbf{1}_A(z, 0)e^{\theta_2 z} \ell(dz), \forall A \in \mathcal{B}(\mathbb{R}^2), \\ \beta_1 &= \begin{pmatrix} -\rho(1 - \beta_2) \\ 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_1(A) = \int_0^\infty \mathbf{1}_A(z, 0) \frac{\rho\beta_2}{\kappa''_L(\theta_2)} z e^{\theta_2 z} \ell(dz), \forall A \in \mathcal{B}(\mathbb{R}^2), \\ \beta_2 &= \begin{pmatrix} 0 \\ -\alpha(1 - \beta_1) \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_2(A) \equiv 0, \forall A \in \mathcal{B}(\mathbb{R}^2). \end{aligned}$$

These characteristics are admissible and correspond to an affine process in $\mathbb{R}_+ \times \mathbb{R}$.

Remark 3.3. Under $Q_{\bar{\theta}, \bar{\beta}}$, σ^2 still satisfies the Langevin equation with different parameters, that is, the measure change preserves the structure of the equations for σ^2 . However, the process L is not a Lévy process under $Q_{\bar{\theta}, \bar{\beta}}$, but it remains a semimartingale. The equation for X is the same under the new measure but with different parameters. Therefore, one can use Itô's Formula again to obtain the following explicit expressions for X and σ^2

$$X(t) = X(s)e^{-\alpha(1-\beta_1)(t-s)} + \frac{\theta_1}{\alpha(1-\beta_1)}(1 - e^{-\alpha(1-\beta_1)(t-s)}) \quad (3.6)$$

$$\begin{aligned}
& + \int_s^t \sigma(u) e^{-\alpha(1-\beta_1)(t-u)} dW_{Q_{\bar{\theta}, \bar{\beta}}}(u), \\
\sigma^2(t) & = \sigma^2(s) e^{-\rho(1-\beta_2)(t-s)} + \frac{\kappa'_L(\theta_2)}{\rho(1-\beta_2)} (1 - e^{-\rho(1-\beta_2)(t-s)}) \\
& + \int_s^t \int_0^\infty e^{-\rho(1-\beta_2)(t-u)} z \tilde{N}_{Q_{\theta, \beta}}^L(du, dz),
\end{aligned} \tag{3.7}$$

where $0 \leq s \leq t \leq T$.

We prove that $Q_{\bar{\theta}, \bar{\beta}}$ is a true probability measure, that is, $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t)$ is a strictly positive true martingale under P for $t \leq T$. We have the following theorem.

Theorem 3.4. *Let $\theta \in \bar{D}_L, \bar{\beta} \in [0, 1]^2$. Then $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2}) = \{\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t)\}_{t \in [0, T]}$ is a strictly positive true martingale under P .*

Proof. That $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})$ is strictly positive follows easily from the fact that the Lévy process L is a subordinator as this yields strictly positive jumps of $\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2}$. It holds that $[\tilde{G}_{\theta_1, \beta_1}, \tilde{H}_{\theta_2, \beta_2}]$, the quadratic co-variation between $\tilde{G}_{\theta_1, \beta_1}$ and $\tilde{H}_{\theta_2, \beta_2}$, is identically zero, by Yor's formula in Protter [21, Theorem 38]. Hence, we can write

$$\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t) = \mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t) \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(t), \quad t \in [0, T]. \tag{3.8}$$

By classical martingale theory, we know that $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})$ is a true martingale if and only if

$$\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(T)] = 1,$$

which, using Yor's formula, is equivalent to showing that

$$\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T) \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(T)] = 1.$$

Let, \mathcal{F}_T^L be the σ -algebra generated by L up to time T , then we have that

$$\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T) \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(T)] = \mathbb{E}_P[\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T) | \mathcal{F}_T^L] \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(T)].$$

If we show that $\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T) | \mathcal{F}_T^L] \equiv 1$, then we will have finished, because by Theorem 3.10 in Benth and Ortiz-Latorre [9], we have that $\mathcal{E}(\tilde{H}_{\theta_2, \beta_2})$ is a true martingale and, hence, $\mathbb{E}_P[\mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(T)] = 1$. The idea of the proof is based on the fact that $\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T) | \mathcal{F}_T^L]$ is the expectation of $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T)$ assuming that $\sigma(t)$ is a deterministic function that, in addition, is bounded below by $\sigma(0)e^{-\rho t}$. Using this information one can show that, conditionally on knowing σ , $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})$ is a true martingale and, hence, $\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(T)] = 1$. Let us sketch the proof that is basically the same as in Section 3.1 in [9] but, now, with σ being a function. First, we show that, conditionally on \mathcal{F}_T^L , $\tilde{G}_{\theta_1, \beta_1}$ is a square integrable P -martingale because

$$\begin{aligned}
\mathbb{E}_P[(\tilde{G}_{\theta_1, \beta_1})^2 | \mathcal{F}_T^L] & = \mathbb{E}_P\left[\int_0^T \sigma^{-2}(t) (\theta_1 + \alpha\beta_1 X(t))^2 dt | \mathcal{F}_T^L\right] \\
& \leq 2\sigma(0)^{-2} e^{2\rho T} \left(\theta_1^2 T + \alpha^2 \mathbb{E}_P\left[\int_0^T X^2(t) dt\right] \right) < \infty,
\end{aligned}$$

(see Proposition 3.6. in [9]). To show that $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})$ is a P -martingale on $[0, T]$, we consider a reducing sequence of stopping times $\{\tau_n\}_{n \geq 1}$ for $\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})$ and, proceeding as in Theorem 3.7 in [9], we define a sequence of probability measure $\{Q_{\theta_1, \beta_1}^n\}_{n \geq 1}$ with Radon-Nykodim densities given by $\frac{dQ_{\theta_1, \beta_1}^n}{dP} \Big|_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1, \beta_1})^{\tau_n}(t), t \in [0, T], n \geq 1$. Doing the same reasonings as in Theorem 3.7 in [9], we reduce the problem to prove that

$$\sup_{n \geq 1} \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]} (G_{\theta_1, \beta_1}(t))^2 dt \right] < \infty.$$

Now, one has

$$\mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]} (G_{\theta_1, \beta_1}(t))^2 dt \right] \leq 2\sigma(0)^{-2} e^{2\rho T} \left(\theta_1^2 T + \alpha^2 \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]} X^2(t) dt \right] \right).$$

To bound the last expectation in the previous expression we use that we know the dynamics of $X(t)$ for $t \in [0, \tau_n]$ under Q_{θ_1, β_1}^n , which is obtained from equation (3.6) by setting $s = 0$ and $t < \tau_n$. Therefore, we can write

$$\begin{aligned} & \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]}(t) X(t)^2 dt \right] \\ & \leq 2 \left\{ \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]}(t) \left(X(0) e^{-\alpha(1-\beta_1)t} + \frac{\theta_1}{\alpha(1-\beta_1)} \left(1 - e^{-\alpha(1-\beta_1)t} \right) \right)^2 dt \right] \right. \\ & \quad \left. + \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\int_0^T \mathbf{1}_{[0, \tau_n]}(t) \left(\int_0^t \sigma(s) e^{-\alpha(1-\beta_1)(t-s)} dW_{Q_{\theta_1, \beta_1}^n}(s) \right)^2 dt \right] \right\} \\ & \leq 2T \{ (|X(0)| + (|\theta_1|) T)^2 + \sigma(0)^{-2} e^{2\rho T} T^2 \} < \infty. \end{aligned}$$

Here, we have used that the function $\eta(x) \triangleq (1 - e^{-xa})/x \leq a$ for $x, a \geq 0$, and that

$$\begin{aligned} & \mathbb{E}_{Q_{\theta_1, \beta_1}^n} \left[\left(\int_0^t \sigma(s) e^{-\alpha(1-\beta_1)(t-s)} dW_{Q_{\theta_1, \beta_1}^n}(s) \right)^2 \right] \\ & = \sigma(0)^{-2} e^{2\rho T} \int_0^t e^{-2\alpha(1-\beta_1)(t-s)} ds \leq \sigma(0)^{-2} e^{2\rho T} T. \end{aligned}$$

The Theorem follows. \square

We also have the following result on the independence of the driving noise processes after the change of measure:

Lemma 3.5. *Under $Q_{\bar{\theta}, \bar{\beta}}$, the Brownian motion $W_{Q_{\bar{\theta}, \bar{\beta}}}$ and the random measure $N_{Q_{\bar{\theta}, \bar{\beta}}}^L$ are independent.*

Proof. To prove the independence of $W_{Q_{\bar{\theta}, \bar{\beta}}}$ and $N_{Q_{\bar{\theta}, \bar{\beta}}}^L$ under $Q_{\bar{\theta}, \bar{\beta}}$, it is sufficient to prove that

$$\begin{aligned} & \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}} \left[\exp \left(i \sum_{j=1}^k \left(\mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right) \right) \right] \\ & = \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}} \left[\exp \left(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) \right) \right] \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}} \left[\exp \left(i \sum_{j=1}^k \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right) \right], \end{aligned}$$

for any $\mu_j, \xi_j \in \mathbb{R}, j = 1, \dots, k$ and $0 \leq t_1 < t_2 < \dots < t_{k-1} < t_k \leq T$. We will make use of the following notation: given a process $Z = \{Z(t)\}_{t \in [0, T]}$ we will denote by $\Delta_j Z = Z(t_j) - Z(t_{j-1}), j = 1, \dots, k$, where $t_0 = 0$, by convention. We have that

$$\begin{aligned} & \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}} \left[\exp \left(i \sum_{j=1}^k \left(\mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right) \right) \right] \\ & = \mathbb{E}_P [\mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t_k) \exp \left(i \sum_{j=1}^k \left(\mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right) \right)] \\ & = \mathbb{E}_P [\mathbb{E}_P [\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_k) \exp \left(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) \right) | \mathcal{F}_k^L] \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(t_k) \exp \left(i \sum_{j=1}^k \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right)], \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_k) \exp(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j)) | \mathcal{F}_{t_k}^L] \\
&= \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_{k-1}) \exp(\int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}(s) dW(s) - \frac{1}{2} \int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}^2(s) ds) \\
&\quad \times \exp(i \sum_{j=1}^{k-1} \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + i \mu_k W_{Q_{\bar{\theta}, \bar{\beta}}}(t_{k-1}) + i \mu_k \Delta W_{Q_{\bar{\theta}, \bar{\beta}}}) | \mathcal{F}_{t_k}^L] \\
&= \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_{k-1}) \exp(i \sum_{j=1}^{k-1} \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + i \mu_k W_{Q_{\bar{\theta}, \bar{\beta}}}(t_{k-1})) \\
&\quad \times \mathbb{E}_P[\exp(\int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}(s) dW(s) - \frac{1}{2} \int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}^2(s) ds \\
&\quad + i \mu_k \left(\Delta_k W - \int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}(s) ds \right)) | \mathcal{F}_{t_k}^L \vee \mathcal{F}_{t_{k-1}}^W] | \mathcal{F}_{t_k}^L].
\end{aligned}$$

Moreover, using similar arguments to those used in the proof of Theorem 3.4, we have that

$$\exp\left(\int_0^t (G_{\theta_1, \beta_1}(s) + i \mu_k) dW(s) - \frac{1}{2} \int_0^t (G_{\theta_1, \beta_1}(s) + i \mu_k)^2 ds\right),$$

is a $\mathcal{F}_{t_k}^L \vee \mathcal{F}_t^W$ -martingale and, then, we get

$$\begin{aligned}
& \mathbb{E}_P[\exp(\int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}(s) dW(s) - \frac{1}{2} \int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}^2(s) ds + i \mu_k \left(\Delta_k W - \int_{t_{k-1}}^{t_k} G_{\theta_1, \beta_1}(s) ds \right)) | \mathcal{F}_{t_k}^L \vee \mathcal{F}_{t_{k-1}}^W] \\
&= e^{-\frac{1}{2} \mu_k^2 \Delta_k t} \mathbb{E}_P[\exp(\int_{t_{k-1}}^{t_k} (G_{\theta_1, \beta_1}(s) + i \mu_k) dW(s) - \frac{1}{2} \int_{t_{k-1}}^{t_k} (G_{\theta_1, \beta_1}(s) + i \mu_k)^2 ds) | \mathcal{F}_{t_k}^L \vee \mathcal{F}_{t_{k-1}}^W] \\
&= e^{-\frac{1}{2} \mu_k^2 \Delta_k t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_k) \exp(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j)) | \mathcal{F}_{t_k}^L] \\
&= e^{-\frac{1}{2} \mu_k^2 \Delta_k t} \mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_{k-1}) \exp(i \sum_{j=1}^{k-1} \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + i \mu_k W_{Q_{\bar{\theta}, \bar{\beta}}}(t_{k-1})) | \mathcal{F}_{t_k}^L].
\end{aligned}$$

Repeating the previous conditioning trick, one gets that

$$\mathbb{E}_P[\mathcal{E}(\tilde{G}_{\theta_1, \beta_1})(t_k) \exp(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j)) | \mathcal{F}_{t_k}^L] = \exp\left(-\frac{1}{2} \sum_{j=1}^k \left(\sum_{q=j}^k \mu_q^2\right) \Delta_j t\right).$$

and, therefore,

$$\begin{aligned}
& \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}}[\exp(i \sum_{j=1}^k \left(\mu_j W_{Q_{\bar{\theta}, \bar{\beta}}}(t_j) + \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz) \right))] \\
&= \exp\left(-\frac{1}{2} \sum_{j=1}^k \left(\sum_{q=j}^k \mu_q\right)^2 \Delta_j t\right) \mathbb{E}_{Q_{\bar{\theta}, \bar{\beta}}}[\exp(i \sum_{j=1}^k \xi_j \int_0^{t_j} \int_0^\infty z \tilde{N}_{Q_{\bar{\theta}, \bar{\beta}}}^L(ds, dz))]
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(i \sum_{j=1}^k \mu_j W_{Q_{\bar{\theta},\bar{\beta}}}(t_j))] &= \mathbb{E}_{Q_{\bar{\theta},\bar{\beta}}}[\exp(i \sum_{j=1}^k \left(\sum_{q=j}^k \mu_q \right) \Delta_j W_{Q_{\bar{\theta},\bar{\beta}}})] \\ &= \exp \left(-\frac{1}{2} \sum_{j=1}^k \left(\sum_{q=j}^k \mu_q \right)^2 \Delta_j t \right),\end{aligned}$$

and we can conclude the proof. \square

One of the particularities of electricity markets is that power is a non storable commodity and for that reason is not a directly tradeable financial asset. This entails that one can not derive the forward price of electricity from the classical buy-and-hold hedging argument. Using a risk-neutral pricing argument (see Benth, Šaltytė Benth and Koekebakker [5]), under the assumption of deterministic interest rates, the forward price at time $0 \leq t$, with time of delivery T with $t \leq T < T^*$, is given by $F_Q(t, T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t]$. Here, Q is any probability measure equivalent to the historical measure P and \mathcal{F}_t is the market information up to time t . In what follows we will use the probability measure $Q = Q_{\bar{\theta},\bar{\beta}}$ introduced above, and let the spot price dynamics be given in terms of the process $X(t)$ and $\sigma^2(t)$ in (2.2)-(2.3). This will provide us with a parametric class of structure-preserving probability measures, extending the Esscher transform but still being reasonably analytically tractable from a pricing point of view.

Our choice of pricing measure can also be applied to temperature futures markets, where the underlying "asset" is a temperature index measured in some location. Temperature is clearly not financially tradeable. There is empirical evidence for mean-reversion and stochastic volatility in temperature data, see Benth and Šaltytė Benth [3]. Yet another example is the freight rate market, where the "spot" typically is an index obtained from opinions of traders. See Benth, Koekebakker and Taib [8] for stochastic modelling of freight rate spot data, with models of the form (2.2)-(2.3).

Oil and gas can typically be stored, and one can build a pricing model for forwards by including storage and transportation costs, as well as the convenience yield (see e.g. Eydeland and Wolyniec [11] and Geman [13]). However, we may also in this case use the probability measure $Q = Q_{\bar{\theta},\bar{\beta}}$ as a parametric class of pricing measures. Firstly, the underlying spot assets do not need to be (local) martingales under the pricing measure in these markets, although the assets are tradable, since there are frictions yielding market incompleteness. Secondly, the probability measures provide a flexible way to model the risk premium (as we shall see later), and therefore may be attractive over models that directly specifies the dynamics of a convenience yield, say (see e.g. Eydeland and Wolyniec [11] for such models). Note that in Benth [2], a model for the spot given by (2.2)-(2.3) has been shown to fit gas prices reasonably well.

We note that in electricity markets, the delivery of the underlying power takes place over a period of time $[T_1, T_2]$, where $0 < T_1 < T_2 < T^*$. We call such contracts swap contracts and we will denote their price at time $t \leq T_1$ by

$$F_Q(t, T_1, T_2) \triangleq \mathbb{E}_Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(T) dT | \mathcal{F}_t \right].$$

We can use the stochastic Fubini theorem to relate the price of forwards and swaps

$$F_Q(t, T_1, T_2) \triangleq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F_Q(t, T) dT.$$

The risk premium for forward prices with a fixed delivery time is defined by the following expression

$$R_Q^F(t, T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t] - \mathbb{E}_P[S(T)|\mathcal{F}_t],$$

and for swap prices by

$$R_Q^S(t, T_1, T_2) \triangleq F_Q(t, T_1, T_2) - \mathbb{E}_Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(T) dT | \mathcal{F}_t \right]$$

It is simple to see that

$$R_Q^S(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} R_Q^F(t, T) dT.$$

The risk premium measures the price discount a producer (seller) of power must accept compared to the predicted spot price at delivery. We shall use the risk premium to analyse the effect of our measure change on forward prices, and to discuss these in relation to stylized facts from the power markets.

4. ARITHMETIC SPOT MODEL

We are interested in applying the previous probability measure change to study the implied risk premium. The first model for the spot price S that we are going to consider is the arithmetic one. We define the *arithmetic spot price model* by

$$S(t) = \Lambda_a(t) + X(t), \quad t \in [0, T^*], \quad (4.1)$$

where $T^* > 0$ is a fixed time horizon. The processes Λ_a is assumed to be deterministic and it accounts for the seasonalities observed in the spot prices. We note in passing that such models have been considered by several authors for various energy markets. We refer to Lucia and Schwartz [20] for power markets and Dornier and Querel [10] for temperature derivatives with no stochastic volatility. More recently Benth, Šaltytė Benth and Koekebakker [5] has a general discussion of arithmetic models in energy markets (see also Garcia, Klüppelberg and Müller [12] for power markets), and Benth, Šaltytė Benth [3] for temperature markets with stochastic volatility.

In order to compute the forward prices and the risk premium associated to them in this model, we need to know the dynamics of S (that is, of X and σ^2) under P and under Q . Explicit expressions for X and σ^2 under P are given by equations (2.4) and (2.5), respectively. In the rest of this section, $Q = Q_{\bar{\theta}, \bar{\beta}}$, $\bar{\theta} \in \bar{D}_L$, $\bar{\beta} \in [0, 1]^2$ defined by (3.3), and the explicit expressions for X and σ^2 under Q are given in Remark 3.3, equations (3.6) and (3.7), respectively.

Proposition 1. *The forward price $F_Q(t, T)$ in the arithmetic spot model 4.1 is given by*

$$F_Q(t, T) = \Lambda_a(T) + X(t)e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)}(1 - e^{-\alpha(1-\beta_1)(T-t)}).$$

Proof. By equation (3.6) and using the basic properties of the conditional expectation we have that

$$\begin{aligned} F_Q(t, T) &= \mathbb{E}_Q[S(T)|\mathcal{F}_t] = \Lambda_a(T) + X(t)e^{-\alpha(1-\beta_1)(T-t)} \\ &\quad + \frac{\theta_1}{\alpha(1-\beta_1)}(1 - e^{-\alpha(1-\beta_1)(T-t)}) \\ &\quad + \mathbb{E}_Q\left[\int_t^T \sigma(s)e^{-\alpha(1-\beta_1)(T-s)} dW_Q(s) | \mathcal{F}_t\right]. \end{aligned}$$

Hence, the proof follows by showing that $\sigma(t)e^{\alpha(1-\beta_1)t}$ belongs to $L^2(\Omega \times [0, T], Q \otimes dt)$ because, then, $\int_0^t \sigma(s)e^{\alpha(1-\beta_1)s} dW_Q(s)$ is a Q -martingale and

$$\mathbb{E}_Q\left[\int_t^T \sigma(s)e^{-\alpha(1-\beta_1)(T-s)} dW_Q(s) | \mathcal{F}_t\right] = e^{-\alpha(1-\beta_1)T} \mathbb{E}_Q\left[\int_t^T \sigma(s)e^{\alpha(1-\beta_1)s} dW_Q(s) | \mathcal{F}_t\right] = 0.$$

Using the dynamics of σ^2 under Q , see equation (3.7), we get

$$\begin{aligned} \mathbb{E}_Q[\sigma^2(t)] &= \sigma^2(0)e^{-\rho(1-\beta_2)t} + \frac{\kappa'_L(\theta_2)}{\rho(1-\beta_2)}(1 - e^{-\rho(1-\beta_2)t}) \\ &\quad + \mathbb{E}_Q\left[\int_0^t \int_0^\infty e^{-\rho(1-\beta_2)(t-s)} z \tilde{N}_Q^L(ds, dz)\right] \\ &\leq \sigma^2(0) + \kappa'_L(\theta_2)t \end{aligned}$$

because $\int_0^t \int_0^\infty e^{-\rho(1-\beta_2)s} z \tilde{N}_Q^L(ds, dz)$ is a Q -martingale starting at 0, see Lemma 4.3 in Benth and Ortiz-Latorre [9]. Hence,

$$\begin{aligned} \mathbb{E}_Q\left[\int_0^T \sigma^2(t) e^{2\alpha t} dt\right] &= \int_0^T \mathbb{E}_Q[\sigma^2(t)] e^{2\alpha t} dt \\ &\leq \int_0^T (\sigma^2(0) + \kappa'_L(\theta_2)t) e^{2\alpha t} dt \\ &\leq T (\sigma^2(0) + \kappa'_L(\theta_2)T) e^{2\alpha T} < \infty, \end{aligned}$$

and we can conclude. \square

Using the previous result on forward prices we get the following formula for the risk premium.

Theorem 4.1. *The risk premium $R_Q^F(t, T)$ for the forward price in the arithmetic spot model (4.1) is given by*

$$R_Q^F(t, T) = X(t) e^{-\alpha(T-t)} \left(e^{\alpha\beta_1(T-t)} - 1 \right) + \frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}).$$

We now analyse the risk premium in more detail under various conditions.

4.1. Discussion on the risk premium. The first remarkable property of this measure change is that it only depends on the parameters that change the speed and level of mean reversion, i.e., θ_1 and β_1 . Moreover, if $\theta_1 = \beta_1 = 0$ we have $R_Q^F(t, T) \equiv 0$, whatever the values of θ_2 and β_2 . This means that, in the arithmetic model, we can have very different pricing measures regarding the volatility properties and have zero risk premium. In other words, there is an unspanned volatility component that can not be explained by just observing the forward curve. Secondly, as long as the parameter $\beta_1 \neq 0$, the risk premium is stochastic. Note that when $\bar{\beta} = (0, 0)$, our measure change coincides with the Esscher transform. In the Esscher case, the risk premium has a deterministic evolution given by

$$R_Q^F(t, T) = \frac{\theta_1}{\alpha} (1 - e^{-\alpha(T-t)}), \quad (4.2)$$

an already known result, see Benth [2].

From now on we shall rewrite the expressions for the risk premium in terms of the time to maturity $\tau = T - t$ and, slightly abusing the notation, we will write $R_Q^F(t, \tau)$ instead of $R_Q^F(t, t + \tau)$. We fix the parameters of the model under the historical measure P , i.e., α and ρ , and study the possible sign of $R_Q^F(t, \tau)$ in terms of the change of measure parameters, i.e., $\bar{\beta} = (\beta_1, \beta_2)$ and $\bar{\theta} = (\theta_1, \theta_2)$ and the time to maturity τ . In fact, we just change θ_1 and β_1 because the risk premium does not depend on the values of θ_2 and β_2 . Note that present time t just enters into the picture through the stochastic component X and not through the volatility process $\sigma^2(t)$. We are going to study the cases $\theta_1 = 0$, $\beta_1 = 0$ and the general case separately. Moreover, in order to graphically illustrate the discussion we plot the risk premium profiles obtained assuming that the subordinator L is a compound Poisson process with jump intensity $c/\lambda > 0$ and exponential jump sizes with mean λ . That is, L will have the Lévy measure given in Example 3.1. We shall measure the time to maturity τ in days and plot $R_Q^F(t, \tau)$ for $\tau \in [0, 360]$, roughly one year. We fix the values of the following parameters

$$\alpha = 0.127, \rho = 1.11, c = 0.4, \lambda = 2.$$

The speed of mean reversion for the factor α yields a half-life of $\log(2)/0.127 = 5.47$ days, while the one for the volatility ρ yields a half-life of $\log(2)/1.11 = 0.65$ days (see e.g., Benth, Šaltytė Benth and Koekebakker [5] for the concept of half-life). The values for c and λ give jumps with mean 0.5 and frequency of 5 spikes in the volatility per month. The values for the speed of mean reversion are obtained from an empirical analysis of the UK gas spot prices conducted in Benth [2].

The following lemma will help in the discussion.

Lemma 4.2. *We have that*

$$R_Q^F(t, \tau) = X(t)e^{-\alpha\tau} \left(e^{\alpha\beta_1\tau} - 1 \right) + \frac{\theta_1}{\alpha(1 - \beta_1)} (1 - e^{-\alpha(1-\beta_1)\tau}).$$

Moreover,

$$\lim_{\tau \rightarrow \infty} R_Q^F(t, \tau) = \frac{\theta_1}{\alpha(1 - \beta_1)}, \quad \text{and} \quad \lim_{\tau \rightarrow 0} \frac{\partial}{\partial \tau} R_Q^F(t, \tau) = X(t)\alpha\beta_1 + \theta_1.$$

Proof. Follows easily from the expression of $R_Q^F(t, \tau)$ in Theorem 4.1. \square

- **Changing the level of mean reversion (Esscher transform):** Setting $\beta_1 = 0$, the probability measure Q only changes the level of mean reversion for the factor X (which is assumed to be zero under the historical measure P). On the other hand, the risk premium is deterministic and cannot change with changing market conditions. From equation (4.2), we get that the sign of $R_Q^F(t, \tau)$ is the same for any time to maturity τ and it is equal to the sign of θ_1 . See Figures 1a and 1b.
- **Changing the speed of mean reversion:** Setting $\theta_1 = 0$, the probability measure Q only changes the speed of mean reversion for the factor X . Note that in this case the risk premium is stochastic and it changes with market conditions. By Lemma 4.2 we have that the risk premium is given by

$$R_Q^F(t, \tau) = X(t)e^{-\alpha\tau} \left(e^{\alpha\beta_1\tau} - 1 \right),$$

with $R_Q^F(t, \tau) \rightarrow 0$ as time to maturity τ tends to infinity. On the other hand, we have that

$$\lim_{\tau \rightarrow 0} \frac{\partial}{\partial \tau} R_Q^F(t, \tau) = X(t)\alpha\beta_1.$$

Hence the risk premium will vanish in the long end of the market. In the short end, it can be both positive or negative and stochastically varying with $X(t)$. See Figure 1c, where the impact of $X(t)$ in the short end is evident as a strongly increasing (from zero) risk premium. A negative value of $X(t)$ would lead to a downward pointing risk premium, before converging to zero.

- **Changing the level and speed of mean reversion simultaneously:** In the general case we can get risk premium profiles with positive values in the short end of the forward curve and negative values in the long end, by choosing $\theta_1 < 0$ but close to zero and β_1 close to 1, assuming that $X(t)$ is positive. See Figure 1d. We recall from Geman [13] that there is empirical and economical evidence for a positive risk premium in the short end of the power forward market, while in the long end one expects the sign of the risk premium to be negative as is the typical situation in commodity forward markets.

Remark 4.3. Note that in order to get a change of sign in the risk premium one must change the level and speed of mean reversion simultaneously, see Figure 1d. It is not possible to get the sign change by using solely the Esscher transform, or only modifying the speed of mean reversion of the factors.

5. GEOMETRIC SPOT MODEL

The second model for the spot price S is the geometric one. We define the *geometric spot price model* by

$$S(t) = \Lambda_g(t) \exp(X(t)), \quad t \in [0, T^*], \quad (5.1)$$

where $T^* > 0$ is a fixed time horizon. The process Λ_g is assumed to be deterministic and it accounts for the seasonalities observed in the spot prices. The forward and the swap contracts are defined analogously to the arithmetic model.

Proposition 2. *The forward price $F_Q(t, T)$ in the geometric spot model (5.1) is given by*

$$F_Q(t, T) = \Lambda_g(T) \exp \left(X(t)e^{-\alpha(1-\beta_1)(T-t)} + \sigma^2(t)e^{-\rho(1-\beta_2)(T-t)} \frac{1 - e^{-(2\alpha - \rho(1-\beta_2))(T-t)}}{2(2\alpha - \rho(1-\beta_2))} \right)$$

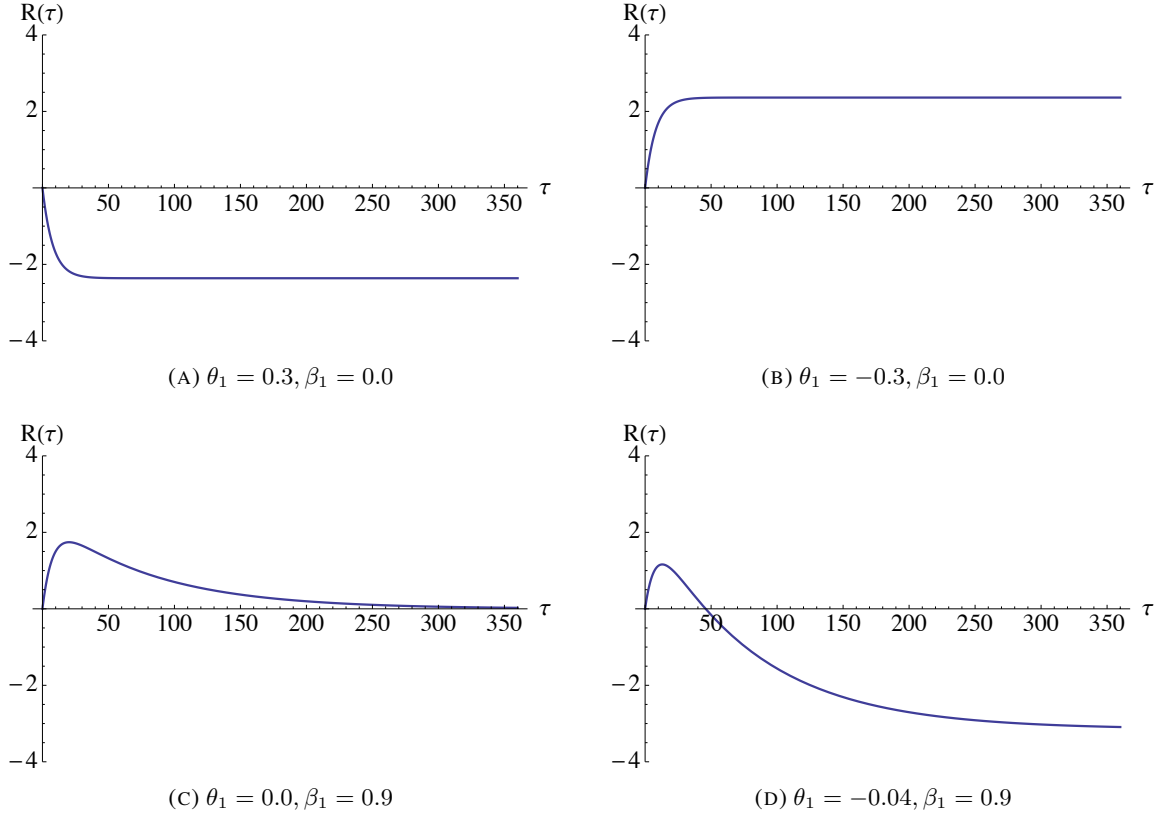


FIGURE 1. Risk premium profiles when L is a Compound Poisson process with exponentially distributed jumps. We take $\rho = 1.11, \alpha = 0.127, \lambda = 2, c = 0.4, X(t) = 2.5, \sigma(t) = 0.25$.

$$\begin{aligned}
& \times \exp \left(\frac{\kappa'_L(\theta_2)}{2\rho(1-\beta_2)} \left(\frac{1-e^{-2\alpha(T-t)}}{2\alpha} - e^{-\rho(1-\beta_2)(T-t)} \frac{1-e^{-(2\alpha-\rho(1-\beta_2))(T-t)}}{(2\alpha-\rho(1-\beta_2))} \right) \right) \\
& \times \exp \left(\frac{\theta_1}{\alpha(1-\beta_1)} (1-e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
& \times \mathbb{E}_Q \left[\exp \left(\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha-\rho(1-\beta_2))s} \left(\int_t^s \int_0^\infty e^{\rho(1-\beta_2)u} z \tilde{N}_Q^L(du, dz) \right) ds \right) | \mathcal{F}_t \right]
\end{aligned}$$

In the particular case $Q = P$, it holds that

$$\begin{aligned}
F_P(t, T) &= \Lambda_g(T) \exp \left(X(t)e^{-\alpha(T-t)} + \sigma^2(t)e^{-\rho(T-t)} \frac{1-e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)} \right) \\
&\quad \times \exp \left(\int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds \right).
\end{aligned}$$

Proof. Denote by \mathcal{F}_t^L the sigma algebra generated by the process L up to time t . Then, we have that

$$\begin{aligned}
\mathbb{E}_Q[S(T)|\mathcal{F}_t] &= \Lambda_g(T) \mathbb{E}_Q[\exp(X(T))|\mathcal{F}_t] \\
&= \Lambda_g(T) \exp \left(X(t)e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)} (1-e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \times \mathbb{E}_Q \left[\exp \left(\int_t^T \sigma(s)e^{-\alpha(1-\beta_1)(T-s)} dW_Q(s) \right) | \mathcal{F}_t \right]
\end{aligned}$$

$$\begin{aligned}
&= \Lambda_g(T) \exp \left(X(t) e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \times \mathbb{E}_Q \left[\mathbb{E}_Q \left[\exp \left(\int_t^T \sigma(s) e^{-\alpha(T-s)} dW_Q(s) \right) \middle| \mathcal{F}_T^L \vee \mathcal{F}_t \right] \middle| \mathcal{F}_t \right] \\
&= \Lambda_g(T) \exp \left(X(t) e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \times \mathbb{E}_Q \left[\exp \left(\frac{1}{2} \int_t^T \sigma^2(s) e^{-2\alpha(T-s)} ds \right) \middle| \mathcal{F}_t \right].
\end{aligned}$$

On the other hand, the dynamics of $\sigma^2(s)$ can be written, for $s > t$, as

$$\begin{aligned}
\sigma^2(s) &= \sigma^2(t) e^{-\rho(1-\beta_2)(s-t)} + \frac{\kappa'_L(\theta_2)}{\rho(1-\beta_2)} (1 - e^{-\rho(1-\beta_2)(s-t)}) \\
&\quad + \int_t^s \int_0^\infty e^{-\rho(1-\beta_2)(s-u)} z \tilde{N}_Q^L(du, dz).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\mathbb{E}_Q[S(T)|\mathcal{F}_t] &= \Lambda_g(T) \exp \left(X(t) e^{-\alpha(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \times \mathbb{E}_Q \left[\exp \left(\frac{1}{2} \int_t^T \left\{ \sigma^2(t) e^{-\rho(1-\beta_2)(s-t)} + \frac{\kappa'_L(\theta_2)}{\rho(1-\beta_2)} (1 - e^{-\rho(1-\beta_2)(s-t)}) \right. \right. \right. \\
&\quad \left. \left. \left. + \int_t^s \int_0^\infty e^{-\rho(1-\beta_2)(s-u)} z N_Q^L(du, dz) \right\} e^{-2\alpha(T-s)} ds \right) \middle| \mathcal{F}_t \right] \\
&= \Lambda_g(T) \exp \left(X(t) e^{-\alpha(1-\beta_1)(T-t)} + \sigma^2(t) e^{-\rho(1-\beta_2)(T-t)} \frac{1 - e^{-(2\alpha - \rho(1-\beta_2))(T-t)}}{2(2\alpha - \rho(1-\beta_2))} \right) \\
&\quad \times \exp \left(\frac{\kappa'_L(\theta_2)}{2\rho(1-\beta_2)} \left(\frac{1 - e^{-2\alpha(T-t)}}{2\alpha} - e^{-\rho(1-\beta_2)(T-t)} \frac{1 - e^{-(2\alpha - \rho(1-\beta_2))(T-t)}}{(2\alpha - \rho(1-\beta_2))} \right) \right) \\
&\quad \times \exp \left(\frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \times \mathbb{E}_Q \left[\exp \left(\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha - \rho(1-\beta_2))s} \left(\int_t^s \int_0^\infty e^{\rho(1-\beta_2)u} z \tilde{N}_Q^L(du, dz) \right) ds \right) \middle| \mathcal{F}_t \right]
\end{aligned}$$

Now, taking into account that $N_Q^L(du, dz)$ has independent increments for $Q = P$, using a stochastic version of Fubini's Theorem and the exponential moments formula for Poisson random measures we obtain

$$\begin{aligned}
&\mathbb{E}_P \left[\exp \left(\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha - \rho)s} \left(\int_t^s \int_0^\infty e^{\rho u} z \tilde{N}^L(du, dz) \right) ds \right) \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_P \left[\exp \left(\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha - \rho)s} \left(\int_t^s \int_0^\infty e^{\rho u} z N^L(du, dz) \right) ds \right) \right] \\
&\quad \times \exp \left(-\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha - \rho)s} \left(\int_t^s \int_0^\infty e^{\rho u} z \ell(dz) du \right) ds \right) \\
&= \mathbb{E}_P \left[\exp \left(\int_t^T \int_0^\infty e^{-\rho(T-u)} \frac{1 - e^{-(2\alpha - \rho(T-u))}}{2(2\alpha - \rho)} z N^L(du, dz) \right) \right] \\
&\quad \times \exp \left(-\frac{e^{-2\alpha T} \kappa'_L(0)}{2\rho} \int_t^T e^{(2\alpha - \rho)s} (e^{\rho s} - e^{\rho t}) ds \right)
\end{aligned}$$

$$\begin{aligned}
&= \exp \left(\int_t^T \int_0^\infty \left(\exp \left(e^{-\rho(T-u)} \frac{1 - e^{-(2\alpha-\rho)(T-u)}}{2(2\alpha-\rho)} z \right) - 1 \right) \ell(dz) du \right) \\
&\quad \times \exp \left(-\frac{e^{-2\alpha T} \kappa'_L(0)}{2\rho} \left(\frac{e^{2\alpha T} - e^{2\alpha t}}{2\alpha} - \frac{e^{2\alpha T} e^{-\rho(T-t)} - e^{2\alpha t}}{2\alpha - \rho} \right) \right) \\
&= \exp \left(\int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds \right) \\
&\quad \times \exp \left(-\frac{\kappa'_L(0)}{2\rho} \left(\frac{1 - e^{-2\alpha(T-t)}}{2\alpha} + \frac{e^{-2\alpha(T-t)} - e^{-\rho(T-t)}}{2\alpha - \rho} \right) \right).
\end{aligned}$$

In the last equality we have used the definition of $\kappa_L(\theta)$ and the change of variable $s = T - u$. Finally, combining the previous expression with the expression for $\mathbb{E}_Q[S(T)|\mathcal{F}_t]$ with $Q = P$, i.e., with $\beta_1 = \beta_2 = \theta_1 = \theta_2 = 0$ we get the result \square

Remark 5.1. Note that $\mathbb{E}_Q[S(T)]$ can be infinite. In the case $Q = P$, if $\Theta_L = \infty$, then $\mathbb{E}_P[S(T)] < \infty$. However, if $\Theta_L < \infty$, then $\mathbb{E}_P[S(T)] < \infty$ if and only if

$$\int_0^T \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds < \infty. \quad (5.2)$$

Condition (5.2) imposes some restrictions on the parameters of the model α, ρ and Θ_L . For $\alpha, \rho > 0$, consider the function $\Upsilon_{\alpha, \rho}(t) = e^{-\rho t} \frac{1 - e^{-(2\alpha-\rho)t}}{2(2\alpha-\rho)}$, $t > 0$. It is easy to see that this function is strictly positive and achieves its maximum at $t^* = -\log(\rho/2\alpha)/(2\alpha - \rho)$ with value $\Upsilon_{\alpha, \rho}(t^*) = \frac{1}{2\rho} \left(\frac{\rho}{2\alpha} \right)^{\frac{1}{1-\frac{\rho}{2\alpha}}}$. Then, it is natural to impose the following assumption on the model parameter that guarantees that condition (5.2) is satisfied for all $T > 0$:

Assumption 2 (\mathcal{P}). We assume that $\alpha, \rho > 0$ and Θ_L satisfy

$$\frac{1}{2\rho} \left(\frac{\rho}{2\alpha} \right)^{\frac{1}{1-\frac{\rho}{2\alpha}}} \leq \Theta_L - \delta,$$

for some $\delta > 0$.

Obviously, if $\Theta_L = \infty$ then assumption \mathcal{P} is satisfied. Suppose that $\Theta_L < \infty$, then if we choose ρ close to zero the value of α must be bounded away from zero, and viceversa, for assumption \mathcal{P} to be satisfied.

The risk premium in the geometric case becomes:

Theorem 5.2. The risk premium $R_Q^F(t, T)$ for the forward price in the geometric spot model (5.1) is given by

$$\begin{aligned}
R_Q^F(t, T) &= \mathbb{E}_P[S(T)|\mathcal{F}_t] \left\{ \exp \left(X(t) e^{-\alpha(T-t)} (e^{\alpha\beta_1(T-t)} - 1) \right) \right. \\
&\quad \times \exp \left(\sigma^2(t) \left(e^{-\rho(1-\beta_2)(T-t)} \frac{1 - e^{-(2\alpha-\rho(1-\beta_2))(T-t)}}{2(2\alpha-\rho(1-\beta_2))} - e^{-\rho(T-t)} \frac{1 - e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)} \right) \right) \\
&\quad \times \exp \left(\frac{\kappa'_L(\theta_2)}{2\rho(1-\beta_2)} \left(\frac{1 - e^{-2\alpha(T-t)}}{2\alpha} - e^{-\rho(1-\beta_2)(T-t)} \frac{1 - e^{-(2\alpha-\rho(1-\beta_2))(T-t)}}{(2\alpha-\rho(1-\beta_2))} \right) \right) \\
&\quad \times \exp \left(\frac{\theta_1}{\alpha(1-\beta_1)} (1 - e^{-\alpha(1-\beta_1)(T-t)}) \right) \\
&\quad \left. \times \exp \left(-\int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds \right) \right\}
\end{aligned}$$

$$\times \mathbb{E}_Q \left[\exp \left(\frac{e^{-2\alpha T}}{2} \int_t^T e^{(2\alpha-\rho)s} \left(\int_t^s \int_0^\infty e^{\rho u} z N_Q^L(du, dz) \right) ds \right) | \mathcal{F}_t \right] - 1 \Big\}$$

Proof. This follows immediately from Proposition 2. \square

The risk premium in the geometric case becomes hard to analyse due to the presence of the conditional expectation in the last term, involving the jump process N_Q^L with respect to Q . In the remainder of this Section we shall rather exploit the affine structure of the model to analyse the risk premium.

5.1. An analysis of the risk premium based on the affine structure. An alternative way of computing $\mathbb{E}_Q[S(T)|\mathcal{F}_t]$, which can provide semi-explicit expressions, is to use the affine structure of $Z = (Z_1(t), Z_2(t))^\top = (\sigma^2(t), X(t))^\top$. Let $\Lambda_i^{\bar{\theta}, \bar{\beta}}(u), i = 0, 1, 2$, be the Lévy exponents associated to the affine characteristics in Remark 3.2, i.e.,

$$\begin{aligned} \Lambda_0^{\bar{\theta}, \bar{\beta}}(u_1, u_2) &= \beta_0^\top u + \frac{1}{2} u^\top \gamma_0 u + \int (e^{u_1 z_1 + u_2 z_2} - 1 - u_1 z_1 - u_2 z_2) \varphi_0(dz) \\ &= \kappa'_L(\theta_2) u_1 + \theta_1 u_2 + \int_0^\infty (e^{u_1 z_1} - 1 - u_1 z_1) e^{\theta_2 z_1} \ell(dz_1) \\ &= \theta_1 u_2 + \kappa_L(u_1 + \theta_2) - \kappa_L(\theta_2), \\ \Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, u_2) &= \beta_1^\top u + u^\top \gamma_1 u + \int (e^{u_1 z_1 + u_2 z_2} - 1 - u_1 z_1 - u_2 z_2) \varphi_1(dz) \\ &= -\rho(1 - \beta_2) u_1 + \frac{u_2^2}{2} + \frac{\rho \beta_2}{\kappa_L''(\theta_2)} \int_0^\infty (e^{u_1 z_1} - 1 - u_1 z_1) z_1 e^{\theta_2 z_1} \ell(dz_1) \\ &= -\rho u_1 + \frac{u_2^2}{2} + \frac{\rho \beta_2}{\kappa_L''(\theta_2)} (\kappa'_L(u_1 + \theta_2) - \kappa'_L(\theta_2)), \\ \Lambda_2^{\bar{\theta}, \bar{\beta}}(u_1, u_2) &= \beta_2^\top u + u^\top \gamma_2 u + \int (e^{u_1 z_1 + u_2 z_2} - 1 - u_1 z_1 - u_2 z_2) \varphi_2(dz) \\ &= -\alpha(1 - \beta_1) u_2. \end{aligned}$$

We find the following:

Theorem 5.3. Let $\bar{\beta} = (\beta_1, \beta_2) \in [0, 1]^2$, $\bar{\theta} = (\theta_1, \theta_2) \in \bar{D}_L$. Assume that there exist functions $\Psi_i^{\bar{\theta}, \bar{\beta}}, i = 0, 1, 2$ belonging to $C^1([0, T]; \mathbb{R}^2)$ satisfying the generalised Riccati equation

$$\begin{aligned} \frac{d}{dt} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) &= -\rho \Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \frac{(\Psi_2^{\bar{\theta}, \bar{\beta}}(t))^2}{2} + \frac{\rho \beta_2}{\kappa_L''(\theta_2)} (\kappa'_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa'_L(\theta_2)), & \Psi_1^{\bar{\theta}, \bar{\beta}}(0) &= 0, \\ \frac{d}{dt} \Psi_2^{\bar{\theta}, \bar{\beta}}(t) &= -\alpha(1 - \beta_1) \Psi_2^{\bar{\theta}, \bar{\beta}}(t), & \Psi_2^{\bar{\theta}, \bar{\beta}}(0) &= 1, \\ \frac{d}{dt} \Psi_0^{\bar{\theta}, \bar{\beta}}(t) &= \theta_1 \Psi_2^{\bar{\theta}, \bar{\beta}}(t) + \kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L(\theta_2), & \Psi_0^{\bar{\theta}, \bar{\beta}}(0) &= 0, \end{aligned} \quad (5.3)$$

and the integrability condition

$$\sup_{t \in [0, T]} \kappa_L''(\theta_2 + \Psi_1^{\bar{\theta}, \bar{\beta}}(t)) < \infty. \quad (5.4)$$

Then,

$$\mathbb{E}_Q[\exp(X(T)) | \mathcal{F}_t] = \exp \left(\Psi_0^{\bar{\theta}, \bar{\beta}}(T-t) + \Psi_1^{\bar{\theta}, \bar{\beta}}(T-t) \sigma^2(t) + \Psi_2^{\bar{\theta}, \bar{\beta}}(T-t) X(t) \right),$$

and

$$\begin{aligned} R_Q^F(t, T) &= \mathbb{E}_P[S(T) | \mathcal{F}_t] \\ &\times \left\{ \exp \left(\Psi_0^{\bar{\theta}, \bar{\beta}}(T-t) - \int_0^{T-t} \kappa_L \left(\frac{e^{-\rho s} (1 - e^{-(2\alpha-\rho)s})}{2(2\alpha-\rho)} \right) ds \right. \right. \\ &\quad \left. \left. + \left(\Psi_1^{\bar{\theta}, \bar{\beta}}(T-t) - e^{-\rho(T-t)} \frac{1 - e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)} \right) \sigma^2(t) \right) \right\} \end{aligned} \quad (5.5)$$

$$+ \left(\Psi_2^{\bar{\theta}, \bar{\beta}}(T-t) - e^{-\alpha(T-t)} \right) X(t) - 1 \Big\}.$$

Proof. The result is a consequence of Theorem 5.1 in Kallsen and Muhle-Karbe [18]: Making the change of variable $t \rightarrow T-t$, the ODE (5.3) is reduced to the one appearing in items 2. and 3. of Theorem 5.1 in Kallsen and Muhle-Karbe [18]. The integrability assumption (5.4) implies conditions 1. and 5., in Theorem 5.1, and condition 4. in that same Theorem is trivially satisfied because $\sigma^2(0)$ and $X(0)$ are deterministic. Hence, the conclusion of Theorem 5.1 in Kallsen and Muhle-Karbe [18], with $p = (0, 1)$, holds and we get

$$\mathbb{E}_Q[\exp(X(T))|\mathcal{F}_t] = \exp \left(\Psi_0^{\bar{\theta}, \bar{\beta}}(T-t) + \Psi_1^{\bar{\theta}, \bar{\beta}}(T-t)\sigma^2(t) + \Psi_2^{\bar{\theta}, \bar{\beta}}(T-t)X(t) \right), \quad t \in [0, T]. \quad (5.6)$$

The result on the risk premium now follows easily. \square

A couple of remarks are in place.

Remark 5.4. The applicability of Theorem 5.3 is quite limited as it is stated. This is due to the fact that it is very difficult to see a priori if there exist functions $\Psi_i^{\bar{\theta}, \bar{\beta}}, i = 0, 1, 2$ belonging to $C^1([0, T]; \mathbb{R}^2)$ satisfying equation (5.3). One has to study existence and uniqueness of solutions of equation (5.3) and the possibility of extending the solution to arbitrary large $T > 0$. We study this problem in Theorem 5.7.

Remark 5.5. Note that the previous system of autonomous ODEs can be effectively reduced to a one dimensional non autonomous ODE. We have that for any $\bar{\theta} \in \bar{D}_L, \bar{\beta} \in [0, 1]^2$, the solution of the second equation is given by $\Psi_2^{\bar{\theta}, \bar{\beta}}(t) = \exp(-\alpha(1-\beta_1)t)$. Plugging this solution to the first equation we get the following equation to solve for $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$

$$\frac{d}{dt} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) = -\rho \Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \frac{e^{-2\alpha(1-\beta_1)t}}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)} (\kappa_L'(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L'(\theta_2)), \quad (5.7)$$

with initial condition $\Psi_1^{\bar{\theta}, \bar{\beta}}(0) = 0$. The equation for $\Psi_0^{\bar{\theta}, \bar{\beta}}(t)$ is solved by integrating $\Lambda_0^{\bar{\theta}, \bar{\beta}}(\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t))$, i.e.,

$$\begin{aligned} \Psi_0^{\bar{\theta}, \bar{\beta}}(t) &= \int_0^t \{ \theta_1 \Psi_2^{\bar{\theta}, \bar{\beta}}(s) + \kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2) \} ds \\ &= \theta_1 \frac{1 - e^{-\alpha(1-\beta_1)t}}{\alpha(1-\beta_1)} + \int_0^t \{ \kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2) \} ds. \end{aligned}$$

As we have already indicated, we cannot in general find the explicit solution of the system of ODEs in Theorem 5.3, and has to rely on numerical techniques. However, the main problem is to ensure the existence and uniqueness of global solutions. Before stating our main result on this question, we introduce some notation and a technical lemma.

Lemma 5.6. Let $\Lambda^{\theta, \beta, a} : [0, \Theta_L - \theta) \rightarrow \mathbb{R}$ be the function defined by

$$\Lambda^{\theta, \beta, a}(u) = -\rho u + a + \frac{\rho\beta}{\kappa_L''(\theta)} (\kappa_L'(u + \theta) - \kappa_L'(\theta)), \quad (5.8)$$

where $a \geq 0, (\theta, \beta) \in D_L \times (0, 1)$ and consider the set

$$\mathcal{D}_b(a) = \{(\theta, \beta) \in D_L \times (0, 1) : \exists u \in [0, \Theta_L - \theta) \text{ s.t. } \Lambda^{\theta, \beta, a}(u) \leq 0\}.$$

Then, we have that:

- (1) For any $(\theta, \beta) \in D_L \times (0, 1)$, there exists a unique global minimum of the function $\Lambda^{\theta, \beta, a}(u)$ which is attained at

$$u^m(\theta, \beta) = (\kappa_L'')^{-1} \left(\frac{\kappa_L''(\theta)}{\beta} \right) - \theta, \quad (5.9)$$

with value

$$\begin{aligned} \Lambda^{\theta,\beta,a}(u^m(\theta, \beta)) &= -\rho \left((\kappa_L'')^{-1} \left(\frac{\kappa_L''(\theta_2)}{\beta_2} \right) - \theta_2 \right) + a \\ &\quad + \left(\frac{\rho\beta_2}{\kappa_L''(\theta_2)} (\kappa_L'((\kappa_L'')^{-1} \left(\frac{\kappa_L''(\theta_2)}{\beta_2} \right))) - \kappa_L'(\theta_2) \right). \end{aligned} \quad (5.10)$$

- (2) The function $\Lambda^{\theta,\beta,a}(u)$ is strictly decreasing in $(0, u^m(\theta, \beta))$ and strictly increasing in $(u^m(\theta, \beta), \Theta_L - \theta)$.
(3) For $\theta \in D_L$ fixed, one has that $u^m(\theta, \beta) \uparrow \Theta_L - \theta$ when $\beta \downarrow 0$ and $u^m(\theta, \beta) \downarrow 0$ when $\beta \uparrow 1$.
(4) The set $\mathcal{D}_b(a)$ coincides with the set

$$\{(\theta, \beta) \in D_L \times (0, 1) : \Lambda^{\theta,\beta,a}(u^m(\theta, \beta)) \leq 0\}.$$

Moreover, for $a > 0$, we have the following:

- (a) If $\theta \in D_L$ is such that $\theta > \Theta_L - a/\rho$ then $\nexists \beta \in (0, 1)$ such that $(\theta, \beta) \in \mathcal{D}_b(a)$.
(b) If $\theta \in D_L$ is such that $\theta < \Theta_L - a/\rho$ then there exists a unique $0 < \beta_m < 1$ such that

$$\Lambda^{\theta,\beta,a}(u^m(\theta, \beta_m)) = 0, \quad (5.11)$$

and for all $\beta \in [0, \beta_m]$ one has $(\theta, \beta) \in \mathcal{D}_b(a)$.

- (5) For $(\theta, \beta) \in \mathcal{D}_b(a)$, $a > 0$ there exists a unique zero of $\Lambda^{\theta,\beta,a}(u)$, denoted by $u_a^0(\theta, \beta)$. As a function of β , $u_a^0(\theta, \beta)$ is well defined on $[0, \beta_m]$, strictly increasing, with $u_a^0(\theta, 0) = a/\rho$ and $u_a^0(\theta, \beta_m) = u^m(\theta, \beta_m)$.

Proof. *Proof of 1.:* According to Remark 2.2 we have that

$$\begin{aligned} \frac{d}{du} \Lambda^{\theta,\beta,a}(u) &= -\rho + \rho\beta \frac{\kappa_L''(u + \theta)}{\kappa_L''(\theta)}, \\ \frac{d^2}{du^2} \Lambda^{\theta,\beta,a}(u) &= \rho\beta \frac{\kappa_L'''(u + \theta)}{\kappa_L''(\theta)} > 0, \end{aligned}$$

which yields that there exists a unique $0 < u^*(\theta, \beta) < \Theta_L - \theta$ for $\theta \in D_L$ and $\beta \in (0, 1)$ such that $\Lambda^{\theta,\beta,a}(u)$ attains a global minimum. In fact, $u^*(\theta_2, \beta_2)$ solves

$$1 = \beta \frac{\kappa_L''(u + \theta)}{\kappa_L''(\theta)}.$$

Moreover, by Remark 2.2 again, we have that $\kappa_L''(u)$ is a strictly increasing function and, hence, it has a well defined inverse $(\kappa_L'')^{-1}(v)$ which yields that $u^m(\theta, \beta)$ and $\Lambda^{\theta,\beta,a}(u^m(\theta, \beta))$ are given by equations (5.9) and (5.10), respectively.

Proof of 2.: It follows from the fact that $\frac{d}{du} \Lambda^{\theta,\beta,a}(u) < 0$, $u \in (0, u^m(\theta_2, \beta_2))$ and $\frac{d}{du} \Lambda^{\theta,\beta,a}(u) > 0$, $(u^m(\theta, \beta), \Theta_L - \theta)$.

Proof of 3.: It follows from the monotonicity of $\kappa_L''(u)$ and the explicit expression of $u^m(\theta, \beta)$ given by equation (5.9). Note that, as a function of β , $u^m(\theta, \beta)$ is a strictly decreasing, continuous function in $(0, 1)$.

Proof of 4. and 5.: For any $\theta \in D_L$, note that

$$a - \rho u \triangleq \Lambda^{\theta,0,a}(u) \leq \Lambda^{\theta,\beta,a}(u), \quad \beta \in (0, 1), u \in [0, \Theta_L - \theta],$$

and $\Lambda^{\theta,0,a}(u) > 0$ if $u \in (0, a/\rho)$. Therefore, if $\theta > \Theta_L - a/\rho$ then $\Lambda^{\theta,\beta,a}(u) > 0$, $u \in [0, \Theta_L - \theta]$ for any $\beta \in (0, 1)$. On the other hand, if $\theta < \Theta_L - a/\rho$ we have that $a/\rho \in [0, \Theta_L - \theta]$. Moreover, defining the function $F(u, \beta) = \Lambda^{\theta,\beta,a}(u)$ and taking into account that $F(a/\rho, 0) = 0$ and,

$$\left. \frac{\partial}{\partial u} F(u, \beta) \right|_{(u,\beta)=(a/\rho,0)} = \left(-\rho + \rho\beta \frac{\kappa_L''(u + \theta)}{\kappa_L''(\theta)} \right) \Big|_{(u,\beta)=(a/\rho,0)} = -\rho < 0,$$

we can apply the implicit function theorem to the equation $F(u, \beta) = 0$ that ensures that there exists a neighborhood U of $(a/\rho, 0)$ in which we can write $u_a^0 = u_a^0(\beta)$, the root of $F(u, \beta) = 0$, as a function of β . Moreover, in U , we have that

$$\begin{aligned} \frac{\partial}{\partial \beta} u_a^0(\beta) &= -\frac{\frac{\partial}{\partial \beta} F(u_a^0(\beta), \beta)}{\frac{\partial}{\partial u} F(u_a^0(\beta), \beta)} = -\frac{\frac{\rho}{\kappa_L''(\theta)}(\theta + \kappa_L'(u_a^0(\beta)) - \kappa_L'(\theta))}{-\rho + \rho\beta \frac{\kappa_L''(u_a^0(\beta) + \theta)}{\kappa_L''(\theta)}} \\ &= \frac{u_a^0(\beta) \int_0^1 \kappa_L''(\theta + \lambda u_a^0(\beta)) d\lambda}{\kappa_L''(\theta) - \beta \kappa_L''(\theta + u_a^0(\beta))}, \end{aligned}$$

which is positive as long as

$$\beta < \frac{\kappa_L''(\theta)}{\kappa_L''(\theta + u_a^0(\beta))}.$$

This yields that $u_a^0(\beta)$ is a well defined and strictly increasing function of β for $\beta \in [0, \beta_m]$, where β_m is the root of equation

$$\Lambda^{\theta, \beta, a}(u^m(\theta, \beta_m)) = 0.$$

Moreover, $u_a^0(0) = a/\rho$ and $u_a^0(\beta_m) = u_a^m(\theta, \beta_m)$. If $\Lambda^{\theta, \beta, a}(u^m(\theta, \beta)) < 0$, the existence of $u^0(\theta, \beta)$ follows from Bolzano's Theorem and the uniqueness from the fact that $\Lambda^{\theta, \beta, a}(u)$ is strictly decreasing in $(0, u^m(\theta, \beta))$. \square

We can now state our main result:

Theorem 5.7. *If $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ and $(\theta_1, \beta_1) \in \mathbb{R} \times [0, 1)$ then $(\Psi_0^{\bar{\theta}, \bar{\beta}}(t), \Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t))$ are $C^1([0, T]; \mathbb{R})$ for any $T > 0$. Moreover,*

$$(\Psi_0^{\bar{\theta}, \bar{\beta}}(t), \Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t)) \longrightarrow \left(\frac{\theta_1}{\alpha(1 - \beta_1)} + \int_0^\infty \{\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L(\theta_2)\} ds, 0, 0 \right), \quad t \rightarrow \infty,$$

and

$$t^{-1} \log \left\| (\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t)) \right\| \rightarrow \gamma, \quad t \rightarrow \infty,$$

where $\gamma = -\alpha(1 - \beta_1)$ or $\gamma = -\rho(1 - \beta_2)$.

Proof. First, recall from Remark 5.5 that the existence and uniqueness of the system of ODEs in Theorem 5.3 can be reduced to establish existence and uniqueness for the one dimensional non autonomous equation (5.7). We have to study the time dependent vector field

$$\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u) \triangleq -\rho u + \frac{e^{-2\alpha(1-\beta_1)t}}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(u + \theta_2) - \kappa_L'(\theta_2)), \quad \bar{\beta} \in (0, 1)^2, \quad \bar{\theta} \in \bar{D}_L.$$

Consider

$$\mathcal{D}(\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}) \triangleq \text{int}(\{u \in \mathbb{R} : \tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u) < \infty\}) = \text{int}(\{u \in \mathbb{R} : \kappa_L'(u + \theta_2) < \infty\}) = (-\infty, \Theta_L - \theta_2),$$

and define

$$\mathcal{D} \triangleq \text{int}\left(\bigcap_{\bar{\beta} \in (0, 1)^2, \bar{\theta} \in \bar{D}_L} \mathcal{D}(\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}})\right) = (-\infty, \Theta_L - \Theta_L/2) = (-\infty, \Theta_L/2).$$

On the other hand, for $u, v \in \mathcal{D}(\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}})$, one has that

$$\left| \tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u) - \tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, v) \right| \leq \rho |u - v| + \frac{\rho\beta_2}{\kappa_L''(\theta_2)} \int_0^\infty |e^{uz} - e^{vz}| z e^{\theta_2 z} \ell(dz),$$

and

$$\int_0^\infty |e^{uz} - e^{vz}| z e^{\theta_2 z} \ell(dz) \leq |u - v| \int_0^\infty e^{(u \vee v + \theta_2)z} z^2 \ell(dz),$$

Moreover, note that

$$\text{int}(\{u \in \mathbb{R} : \int_0^\infty z^2 e^{(u + \theta_2)z} \ell(dz) < \infty\}) = (-\infty, \Theta_L - \theta_2) = \mathcal{D}(\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}).$$

Hence, the vector field $\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u)$, $\bar{\theta} \in \bar{D}_L$, $\bar{\beta} \in [0, 1]^2$ is well defined (i.e., finite) and locally Lipschitz in $\mathcal{D}(\tilde{\Lambda}_1)$. Then, by the Picard-Lindelöf Theorem (see Theorem 3.1, page 18, in Hale [16]) we have local existence and uniqueness for $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$ with $\Psi_1^{\bar{\theta}, \bar{\beta}}(0) = 0 \in \mathcal{D}(\tilde{\Lambda}_1)$.

Let us consider the autonomous vector field

$$\hat{\Lambda}_1^{\theta_2, \beta_2}(u) \triangleq -\rho u + \frac{1}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(u + \theta_2) - \kappa_L'(\theta_2)), \quad \beta_2 \in (0, 1), \quad \theta_2 \in D_L.$$

Then, as $\hat{\Lambda}_1^{\theta_2, \beta_2}(u) - \tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u) = \frac{1}{2}(1 - e^{-2\alpha t}) \geq 0$, for $u \geq 0$, using a comparison theorem we have that the solution for the ODE associated to $\tilde{\Lambda}_1^{\bar{\theta}, \bar{\beta}}(t, u)$ and starting at 0 is bounded above by the corresponding solution to the ODE associated to $\hat{\Lambda}_1^{\theta_2, \beta_2}(u)$, which we will denote by $\hat{\Psi}_1^{\theta_2, \beta_2}(t)$. By Lemma 5.6, if $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ there exists a unique $2/\rho < u_{1/2}^0(\theta_2, \beta_2) \leq u^m(\theta_2, \beta_2)$ such that $\hat{\Lambda}_1^{\theta, \beta}(u) > 0$, for $u \in (0, u_{1/2}^0(\theta_2, \beta_2))$ and $\hat{\Lambda}_1^{\theta, \beta}(u_{1/2}^0(\theta_2, \beta_2)) = 0$. This yields that the solution $\hat{\Psi}_1^{\theta_2, \beta_2}(t)$ is defined for $t \in [0, +\infty)$ and monotonously converges to $u_{1/2}^0(\theta_2, \beta_2)$, which is a stationary point of $\hat{\Lambda}_1^{\theta, \beta}(u)$. Hence, the solution $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$ is bounded by $u_{1/2}^0(\theta_2, \beta_2)$ and defined for $t \in [0, +\infty)$. To prove that actually $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$ converges to zero it is convenient to look at the 2 dimensional system

$$\begin{aligned} \frac{d}{dt}\Psi_1^{\bar{\theta}, \bar{\beta}}(t) &= -\rho\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \frac{(\Psi_2^{\bar{\theta}, \bar{\beta}}(t))^2}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L'(\theta_2)), & \Psi_1^{\bar{\theta}, \bar{\beta}}(0) &= 0, \\ \frac{d}{dt}\Psi_2^{\bar{\theta}, \bar{\beta}}(t) &= -\alpha(1 - \beta_1)\Psi_2^{\bar{\theta}, \bar{\beta}}(t), & \Psi_2^{\bar{\theta}, \bar{\beta}}(0) &= 1, \end{aligned} \quad (5.12)$$

with the corresponding vector fields

$$\begin{aligned} \Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, u_2) &= -\rho u_1 + \frac{u_2^2}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(u_1 + \theta_2) - \kappa_L'(\theta_2)), \\ \Lambda_2^{\bar{\theta}, \bar{\beta}}(u_1, u_2) &= -\alpha(1 - \beta_1)u_2. \end{aligned}$$

Note that $(u_1, u_2) = (0, 0)$ is a stationary point, that $\Lambda_1^{\bar{\theta}, \bar{\beta}}(0, u_2) > 0$ for $u_2 > 0$, that $\Lambda_2^{\bar{\theta}, \bar{\beta}}(u_1, 0) = 0$ for $u_1 > 0$ and $\Lambda_2^{\bar{\theta}, \bar{\beta}}(u_1, u_2) < 0$ for $u_1 > 0, u_2 > 0$. Hence, the region $S_{\bar{\theta}, \bar{\beta}} = \{(u_1, u_2) : 0 \leq u_1 < \Theta_L - \theta_2, 0 \leq u_2 \leq 1\}$ is invariant for this vector field, i.e., a solution that enters $S_{\bar{\theta}, \bar{\beta}}$ cannot leave $S_{\bar{\theta}, \bar{\beta}}$. Moreover, we have that the vector field $\Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, u_2)$ evaluated at the line $u_2 = 0$ has the form

$$\Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, 0) = -\rho u_1 + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(u_1 + \theta_2) - \kappa_L'(\theta_2)),$$

i.e., $\Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, 0) = \Lambda^{\theta_2, \beta_2, 0}(u_1)$. By Lemma 5.6, it then follows that $\Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, 0) = \Lambda^{\theta_2, \beta_2, 0}(u_1) < 0$, for $u_1 \in (0, u^m(\theta_2, \beta_2))$. In addition, if $(\theta_2, \beta_2) \in \mathcal{D}(1/2)$, we have that $u_{1/2}^0(\theta_2, \beta_2) < u^m(\theta_2, \beta_2)$. This means that $\Lambda_1^{\bar{\theta}, \bar{\beta}}(u_1, 0) < 0$ for $u_1 \in (0, u_{1/2}^0(\theta_2, \beta_2))$, which can be extended to $(u_1, u_2) \in (0, u_{1/2}^0(\theta_2, \beta_2)) \times (0, \delta) \triangleq R_{\bar{\theta}, \bar{\beta}}(\delta)$ for some $0 < \delta < 1$. Note that $R_{\bar{\theta}, \bar{\beta}}(\delta)$ is in the domain of attraction of the stationary point $(0, 0)$. As $\Psi_2^{\bar{\theta}, \bar{\beta}}(t) = e^{-\alpha(1-\beta_1)t}$ and $\Psi_1^{\bar{\theta}, \bar{\beta}}(t) < u_{1/2}^0(\theta_2, \beta_2)$, we have that $(\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t)) \in \text{int}(R_{\bar{\theta}, \bar{\beta}}(\delta))$ for $t > -\frac{\log(\delta)}{\alpha(1-\beta_1)}$ and, hence, it converges to $(0, 0)$ when t tends to infinity. Note that we can look at the system (5.12) as a perturbed linear system, i.e.,

$$\begin{aligned} \frac{d}{dt}\Psi_1^{\bar{\theta}, \bar{\beta}}(t) &= -\rho(1 - \beta_2)\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + G_1(\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t)), & \Psi_1^{\bar{\theta}, \bar{\beta}}(0) &= 0, \\ \frac{d}{dt}\Psi_2^{\bar{\theta}, \bar{\beta}}(t) &= -\alpha(1 - \beta_1)\Psi_2^{\bar{\theta}, \bar{\beta}}(t) + G_2(\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t)), & \Psi_2^{\bar{\theta}, \bar{\beta}}(0) &= 1, \end{aligned}$$

where

$$\begin{aligned} G_1(u_1, u_2) &= \frac{u_2^2}{2} + \frac{\rho\beta_2}{\kappa_L''(\theta_2)}\left(\int_0^1 \int_0^1 \kappa_L'''(\theta_2 + \lambda_1\lambda_2 u_1) d\lambda_2 \lambda_1 d\lambda_1\right) u_1^2, \\ G_2(u_1, u_2) &= 0, \end{aligned}$$

and

$$\lim_{(u_1, u_2) \rightarrow (0,0)} \frac{(G_1(u_1, u_2), G_2(u_1, u_2))}{\sqrt{u_1^2 + u_2^2}} = (0, 0).$$

Hence, that $(\Psi_1^{\bar{\theta}, \bar{\beta}}(t), \Psi_2^{\bar{\theta}, \bar{\beta}}(t))$ converges to zero exponentially fast follows from Theorem 3.1.(i), Chapter VII, in Hartman [17]. On the other hand, by Remark 5.5 and the monotone convergence theorem, we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} \Psi_0^{\bar{\theta}, \bar{\beta}}(t) &= \lim_{t \rightarrow \infty} \theta_1 \frac{1 - e^{-\alpha(1-\beta_1)t}}{\alpha(1-\beta_1)} + \int_0^t \{\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2)\} ds \\ &= \frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \{\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2)\} ds < \infty. \end{aligned}$$

To prove that the previous integral is finite, note first that, as $\Psi_1^{\bar{\theta}, \bar{\beta}}(t) < u_{1/2}^0(\theta_2, \beta_2)$ and the function $\kappa_L(u)$ is increasing we have that $\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) \leq \kappa_L(u_{1/2}^0(\theta_2, \beta_2) + \theta_2)$. But, by definition

$$0 = -\rho u_{1/2}^0(\theta_2, \beta_2) + \frac{1}{2} + \frac{\rho \beta_2}{\kappa_L''(\theta_2)} (\kappa_L'(u_{1/2}^0(\theta_2, \beta_2) + \theta_2) - \kappa_L'(\theta_2)),$$

which yields that

$$\kappa_L'(u_{1/2}^0(\theta_2, \beta_2) + \theta_2) = \frac{\kappa_L''(\theta_2)}{\rho \beta_2} (\rho u_{1/2}^0(\theta_2, \beta_2) - \frac{1}{2}) + \kappa_L'(\theta_2),$$

which is bounded. Hence, it suffices to prove that $\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L(\theta_2)$ converges to zero faster than $t^{-(1+\varepsilon)}$, for some $\varepsilon > 0$, when t tends to infinity. We have that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{(1+\varepsilon)} (\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2) - \kappa_L(\theta_2)) &= \lim_{t \rightarrow \infty} t^{(1+\varepsilon)} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) \int_0^1 \kappa_L'(\theta_2 + \lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(t)) d\lambda \\ &= \left(\lim_{t \rightarrow \infty} t^{(1+\varepsilon)} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) \right) \left(\lim_{t \rightarrow \infty} \int_0^1 \kappa_L'(\theta_2 + \lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(t)) d\lambda \right) \\ &= \kappa_L'(\theta_2) \lim_{t \rightarrow \infty} t^{(1+\varepsilon)} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) = 0, \end{aligned}$$

because $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$ converges to zero exponentially fast and $\lim_{t \rightarrow \infty} \int_0^1 \kappa_L'(\theta_2 + \lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(t)) d\lambda = \kappa_L'(\theta_2)$ by bounded convergence. \square

An immediate consequence of the Theorem above is that the forward price will be equal to the seasonal function $\Lambda_g(T)$ in the long end, that is, when $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ and $(\theta_1, \beta_1) \in \mathbb{R} \times [0, 1)$, it holds that

$$\lim_{T \rightarrow \infty} \frac{F_Q(t, T)}{\Lambda_g(T)} = \exp \left(\frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \{\kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2)\} ds \right).$$

Note that to have this limiting de-seasonalized forward price, we must compute an integral of a nonlinear function of $\Psi_1^{\bar{\theta}, \bar{\beta}}(t)$, for which we do not have any explicit solution available. Note that from part 4(b) in Lemma 5.6 we have $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ if $\theta_2 < \Theta_L - 1/2\rho$ and $\beta_2 \in [0, \beta_m]$, for a uniquely defined $0 < \beta_m < 1$. We recall that ρ is the speed of mean reversion of the stochastic volatility $\sigma^2(t)$, and Θ_L is the maximal exponential integrability of L , the subordinator driving the same process. Thus, we must choose θ_2 less than Θ_L , by a distance given by the inverse of the speed of mean reversion. Then we know there exists an interval of β_2 's for which we can reduce the speed of mean reversion of $\sigma^2(t)$. Here we see clearly the competition between the jumps of L and the speed of mean reversion of $\sigma^2(t)$.

We note that if we just change the levels of mean reversion, that is assuming $\bar{\beta} = (0, 0)$, then we can compute the risk premium more explicitly. This case will correspond to an Esscher transform of both the Brownian motion driving X and the subordinator L driving $\sigma^2(t)$.

Proposition 3. Suppose that $\bar{\beta} = (0, 0)$ and $\bar{\theta} \in \mathbb{R} \times D_L$. Then the forward price is given by

$$\begin{aligned} \mathbb{E}_Q[\exp(X(T))|\mathcal{F}_t] = \exp \left(e^{-\rho(T-t)} \frac{1 - e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)} \sigma^2(t) + e^{-\alpha(T-t)} X(t) \right. \\ \left. + \theta_1 \frac{1 - e^{-\alpha(T-t)}}{\alpha} + \int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} + \theta_2 \right) - \kappa_L(\theta_2) ds \right), \end{aligned}$$

and the risk premium by

$$\begin{aligned} R_Q^F(t, T) = \mathbb{E}_P[S(T)|\mathcal{F}_t] \left\{ \exp \left(\int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} + \theta_2 \right) - \kappa_L(\theta_2) ds \right. \right. \\ \left. \left. - \int_0^{T-t} \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds + \theta_1 \frac{1 - e^{-\alpha(T-t)}}{\alpha} \right) - 1 \right\}. \end{aligned} \quad (5.13)$$

Proof. Note that the system of generalised Riccati equations to solve is

$$\begin{aligned} \frac{d}{dt} \Psi_1^{\bar{\theta},0}(t) &= -\rho \Psi_1^{\bar{\theta},0}(t) + \frac{(\Psi_2^{\bar{\theta},0}(t))^2}{2} & \Psi_1^{\bar{\theta},0}(0) &= 0, \\ \frac{d}{dt} \Psi_2^{\bar{\theta},0}(t) &= -\alpha \Psi_2^{\bar{\theta},0}(t), & \Psi_2^{\bar{\theta},0}(0) &= 1, \\ \frac{d}{dt} \Psi_0^{\bar{\theta},0}(t) &= \theta_1 \Psi_2^{\bar{\theta},0}(t) + \kappa_L(\Psi_1^{\bar{\theta},0}(t) + \theta_2) - \kappa_L(\theta_2), & \Psi_0^{\bar{\theta},0}(0) &= 0. \end{aligned}$$

With respect to $\Psi_1^{\bar{\theta},0}(t)$ and $\Psi_2^{\bar{\theta},0}(t)$, this coincides with the one satisfied by $\Psi_1^{0,0}(t) = e^{-\rho t} \frac{1 - e^{-(2\alpha-\rho)t}}{2(2\alpha-\rho)}$ and $\Psi_2^{0,0}(t) = e^{-\alpha t}$. Hence, $\Psi_1^{\bar{\theta},0}(t) = e^{-\rho t} \frac{1 - e^{-(2\alpha-\rho)t}}{2(2\alpha-\rho)}$, $\Psi_2^{\bar{\theta},0}(t) = e^{-\alpha t}$ and we just need to integrate the equation for $\Psi_0^{\bar{\theta},0}(t)$ to obtain that

$$\begin{aligned} \Psi_0^{\bar{\theta},0}(t) &= \theta_1 \int_0^t \Psi_2^{\bar{\theta},0}(s) ds + \int_0^t \kappa_L(\Psi_1^{\bar{\theta},0}(s) + \theta_2) - \kappa_L(\theta_2) \\ &= \theta_1 \frac{1 - e^{-\alpha t}}{\alpha} + \int_0^t \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} + \theta_2 \right) - \kappa_L(\theta_2) ds, \end{aligned}$$

to conclude. \square

Next, we present two examples where we apply the previous results.

Example 5.8. We start by the simplest possible case. Assume that the Lévy measure is $\delta_{\{1\}}(dz)$, that is, the Lévy process L has only jumps of size 1. In this case $\Theta_L = \infty$ and, hence, $D_L = \mathbb{R}$. We have that $\kappa_L(\theta_2) = e^{\theta_2} - 1$ and $\kappa_L^{(n)}(\theta_2) = e^{\theta_2}$, $n \in \mathbb{N}$. Therefore, the associated generalised Riccati equation is given

$$\begin{aligned} \frac{d}{dt} \Psi_1^{\bar{\theta},\bar{\beta}}(t) &= -\rho \Psi_1^{\bar{\theta},\bar{\beta}}(t) + \frac{(\Psi_2^{\bar{\theta},\bar{\beta}}(t))^2}{2} + \rho \beta_2 (e^{\Psi_1^{\bar{\theta},\bar{\beta}}(t)} - 1), & \Psi_1^{\bar{\theta},\bar{\beta}}(0) &= 0, \\ \frac{d}{dt} \Psi_2^{\bar{\theta},\bar{\beta}}(t) &= -\alpha(1 - \beta_1) \Psi_2^{\bar{\theta},\bar{\beta}}(t), & \Psi_2^{\bar{\theta},\bar{\beta}}(0) &= 1, \\ \frac{d}{dt} \Psi_0^{\bar{\theta},\bar{\beta}}(t) &= \theta_1 \Psi_2^{\bar{\theta},\bar{\beta}}(t) + e^{\theta_2} (e^{\Psi_1^{\bar{\theta},\bar{\beta}}(t)} - 1), & \Psi_0^{\bar{\theta},\bar{\beta}}(0) &= 0, \end{aligned} \quad (5.14)$$

In this example

$$\hat{\Lambda}_1^{\theta_2, \beta_2}(u) = \Lambda^{\theta_2, \beta_2, 1/2}(u) = \frac{1}{2} - \rho u + \rho \beta_2 (e^u - 1),$$

which does not depend on θ_2 . By Lemma 5.6, $\Lambda^{\theta, \beta, 1/2}(u)$ attains its minimum at

$$u_{1/2}^m(\theta_2, \beta_2) = \log \left(\frac{e^{\theta_2}}{\beta_2} \right) - \theta_2 = -\log(\beta_2)$$

and equation (5.11) reads

$$\Lambda^{\theta_2, \beta_2, 1/2}(u_{1/2}^m) = \frac{1}{2} + \rho \log(\beta_2) + \rho(1 - \beta_2) = 0. \quad (5.15)$$

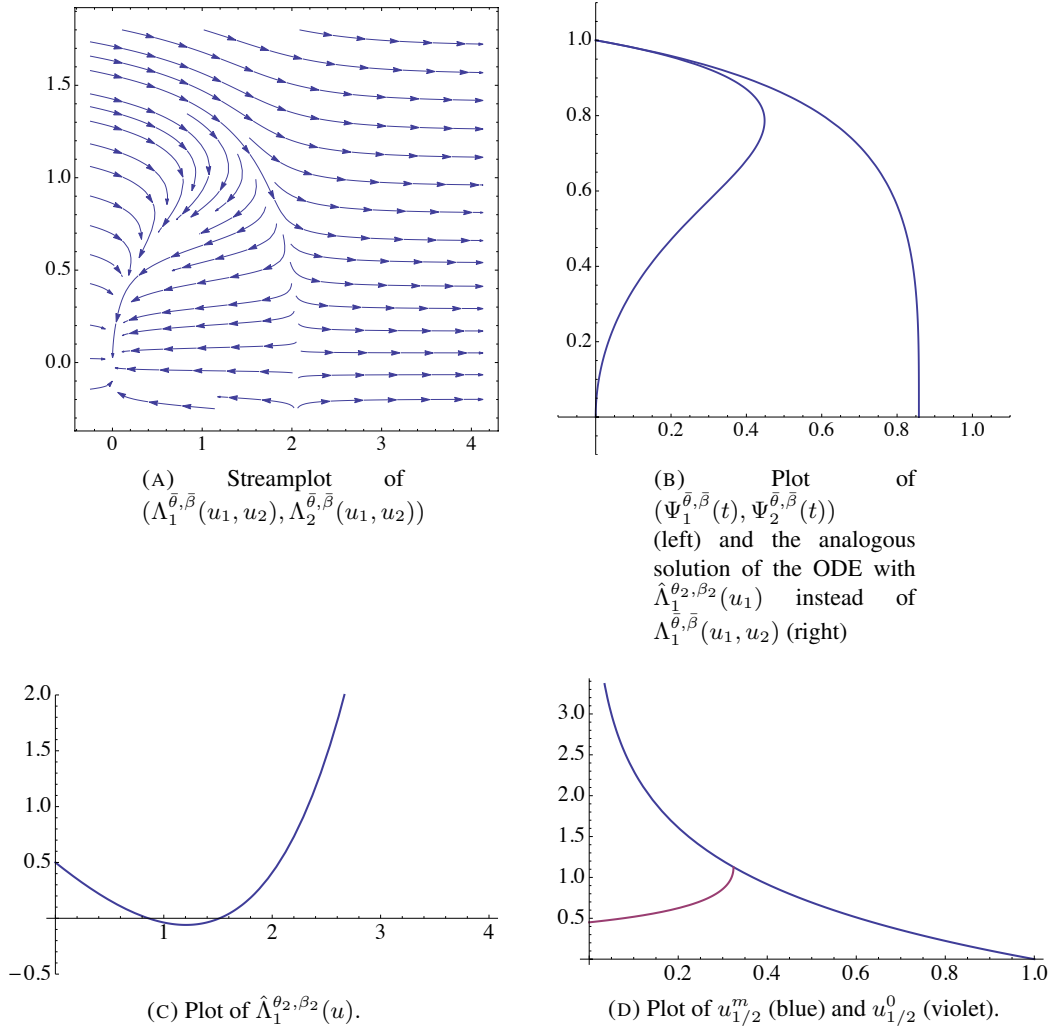


FIGURE 2. Some plots related to example 5.8. We take $\rho = 1.11, \alpha = 0.127, \beta_1 = \beta_2 = 0.3$.

Using the Lambert W function, i.e., the function defined by $W(z)e^{W(z)} = z, z \in \mathbb{C}$, we get that β_m , the root of equation (5.15) is given by

$$\beta_m = -W(-e^{-(1+\frac{1}{2\rho})}).$$

Hence, according to Lemma 5.6, the set $\mathcal{D}_b(\frac{1}{2}) = \{(\theta_2, \beta_2) : \beta_2 \in [0, \beta_m]\}$ and if $\beta_2 \in [0, \beta_m]$ there exists a unique root $u_{1/2}^0(\theta_2, \beta_2)$ of equation $\Lambda^{\theta_2, \beta_2, 1/2}(u) = 0$ satisfying $u_{1/2}^0(\theta_2, \beta_2) \leq u_{1/2}^m(\theta_2, \beta_2)$. This root is given by

$$u_{1/2}^0(\beta_2) = \frac{1}{2\rho} - \left(\beta_2 + W(-\beta_2 e^{(\frac{1}{2\rho} - \beta_2)}) \right).$$

See Figure 2 for a graphical illustration of this case.

Example 5.9. Assume that the Lévy measure is $\ell(dz) = ce^{-\lambda z} \mathbf{1}_{(0, \infty)}$, that is, L is a compound Poisson process with intensity c/λ and exponentially distributed jumps with mean $1/\lambda$. In this case $\Theta_L = \lambda$ and, hence, $D_L = (-\infty, \lambda/2)$. We have that $\kappa_L(\theta) = c\theta/\lambda(\lambda - \theta)$ and $\kappa_L^{(n)}(\theta) = cn!/(\lambda - \theta)^{n+1}, n \in \mathbb{N}$.

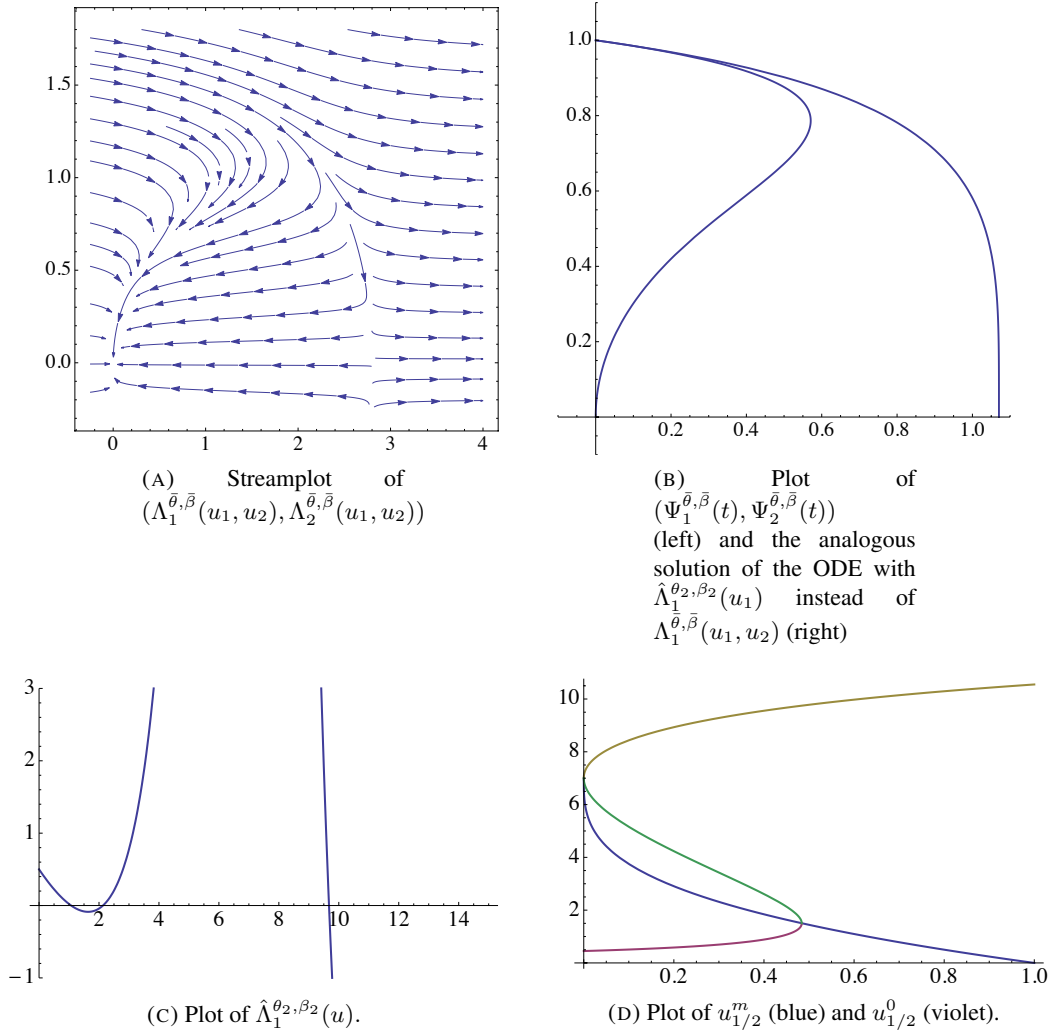


FIGURE 3. Some plots related to example 5.9. We take $\rho = 1.11, \alpha = 0.127, \lambda = 2, \theta_1 \in \mathbb{R}, \theta_2 = -5, \beta_1 = \beta_2 = 0.45$.

Therefore, the associated generalised Riccati equation is given by

$$\begin{aligned}
 \frac{d}{dt} \Psi_1^{\bar{\theta}, \bar{\beta}}(t) &= -\rho \Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \frac{(\Psi_2^{\bar{\theta}, \bar{\beta}}(t))^2}{2} + \frac{\rho \beta_2 (\lambda - \theta_2)^3}{2} \left\{ \frac{1}{(\lambda - \theta_2 - \Psi_1^{\bar{\theta}, \bar{\beta}}(t))^2} - \frac{1}{(\lambda - \theta_2)^2} \right\}, & \Psi_1^{\bar{\theta}, \bar{\beta}}(0) &= 0, \\
 \frac{d}{dt} \Psi_2^{\bar{\theta}, \bar{\beta}}(t) &= -\alpha(1 - \beta_1) \Psi_2^{\bar{\theta}, \bar{\beta}}(t), & \Psi_2^{\bar{\theta}, \bar{\beta}}(0) &= 1, \\
 \frac{d}{dt} \Psi_0^{\bar{\theta}, \bar{\beta}}(t) &= \theta_1 \Psi_2^{\bar{\theta}, \bar{\beta}}(t) + \frac{c(\Psi_1^{\bar{\theta}, \bar{\beta}}(t) + \theta_2)}{\lambda(\lambda - \theta_2 - \Psi_1^{\bar{\theta}, \bar{\beta}}(t))} - \frac{c\theta_2}{\lambda(\lambda - \theta_2)}, & \Psi_0^{\bar{\theta}, \bar{\beta}}(0) &= 0,
 \end{aligned} \tag{5.16}$$

In this example,

$$\hat{\Lambda}_1^{\theta_2, \beta_2}(u) = \Lambda^{\theta_2, \beta_2, 1/2}(u) = \frac{1}{2} - \rho u + \frac{\rho \beta_2 (\lambda - \theta_2)^3}{2} \left\{ \frac{1}{(\lambda - \theta_2 - u)^2} - \frac{1}{(\lambda - \theta_2)^2} \right\}.$$

By Lemma 5.6, $\Lambda^{\theta, \beta, 1/2}(u) : [0, \lambda - \theta_2] \rightarrow \mathbb{R}$ attains its minimum at

$$u_{1/2}^m(\theta_2, \beta_2) = (\lambda - \theta_2) \left(1 - \beta_2^{1/3} \right)$$

and equation (5.11) reads

$$\Lambda^{\theta_2, \beta_2, 1/2}(u_{1/2}^m(\theta_2, \beta_2)) = \frac{1}{2} - \rho(\lambda - \theta_2) + \frac{3}{2}\rho(\lambda - \theta_2)\beta_2^{1/3} - \frac{1}{2}\rho(\lambda - \theta_2)\beta_2 = 0. \quad (5.17)$$

According to Lemma 5.6, if $\theta_2 > \lambda - \frac{1}{2\rho}$ then $\exists \beta_2 \in (0, 1)$ such that $(\theta_2, \beta_2) \in \mathcal{D}_b(\frac{1}{2})$. If $\theta_2 < \lambda - \frac{1}{2\rho}$ then there exists $\beta_m \in (0, 1)$ such that $(\theta_2, \beta_2) \in \mathcal{D}_b(\frac{1}{2})$ for all $\beta_2 \in (0, \beta_m)$ and β_m is the unique solution of equation (5.17) lying in $(0, 1)$. Making the change of variable $z = \beta^{1/3}$, equation (5.17) is reduced to a cubic equation and we get

$$\beta_m = \left(\left(\frac{2}{\sqrt{a(\lambda, \rho, \theta_2)^2 - 4} - a(\lambda, \rho, \theta_2)} \right)^{1/3} + \left(\frac{\sqrt{a(\lambda, \rho, \theta_2)^2 - 4} - a(\lambda, \rho, \theta_2)}{2} \right)^{1/3} \right)^3,$$

where

$$a(\lambda, \rho, \theta_2) \triangleq \frac{2\rho(\lambda - \theta_2) - 1}{\rho(\lambda - \theta_2)} > 0.$$

Finally, if $(\theta_2, \beta_2) \in \mathcal{D}_b(\frac{1}{2})$ there exists a unique root $u_{1/2}^0(\beta_2)$ of equation $\Lambda^{\theta_2, \beta_2, 1/2}(u) = 0$ satisfying $u_{1/2}^0(\theta_2, \beta_2) \leq u_{1/2}^m(\theta_2, \beta_2)$. Making the change of variable $y = \frac{\lambda - \theta_2}{\lambda - \theta_2 - u}$, we can reduce the equation $\Lambda^{\theta_2, \beta_2, 1/2}(u) = 0$ to the cubic equation

$$P_3(y) \triangleq \beta_2 y^3 - (a(\lambda, \rho, \theta_2) + \beta_2)y + 2 = 0,$$

which can be solved explicitly. Inverting the change of variable, we can get an explicit expression for $u_{1/2}^0(\theta_2, \beta_2)$. We refrain to write this explicit formula because it is too lengthy.

See Figure 3 for some graphical illustrations of this example.

5.2. Discussion on the risk premium. The next step is to analyse qualitatively the possible risk premium profiles that can be obtained using our change of measure. In particular, we are interested to be able to generate risk profiles with positive values in the short end of the forward curve and negative values in the long end. In what follows we shall make use of the Musiela parametrization $\tau = T - t$ and we will slightly abuse the notation by denoting $R_Q^F(t, T)$ by $R_Q^F(t, \tau)$. We also fix the parameters of the model under the historical measure P , i.e., α and ρ , and study the possible sign of $R_Q^F(t, \tau)$ in terms of the change of measure parameters, i.e., $\bar{\beta} = (\beta_1, \beta_2)$ and $\bar{\theta} = (\theta_1, \theta_2)$ and the time to maturity τ .

In contrast to the arithmetic model, the present time enters into the risk premium not only through the stochastic components X , but also through the stochastic volatility σ^2 . Moreover, in the geometric model, the risk premium will also depend on the parameters θ_2 and β_2 , which change the level and speed of mean reversion for the volatility process. We are going to study the cases $\bar{\theta} = (0, 0)$, $\bar{\beta} = (0, 0)$ and the general case separately. Moreover, in order to graphically illustrate the discussion we plot the risk premium profiles obtained assuming that the subordinator L is a compound Poisson process with jump intensity $c/\lambda > 0$ and exponential jump sizes with mean λ . That is, L will have the Lévy measure given in Example 3.1. We shall measure the time to maturity τ in days and plot $R_Q^F(t, \tau)$ for different maturity periods. We fix the parameters of the model under the historical measure P using the same values as in the arithmetic case, i.e.,

$$\alpha = 0.127, \rho = 1.11, c = 0.4, \lambda = 2.$$

Finally, in the sequel, we are going to suppose that we are under the assumptions of Theorem 5.7, i.e., the values θ_2, β_2 are such that $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ and $\Psi_0^{\bar{\theta}, \bar{\beta}}, \Psi_1^{\bar{\theta}, \bar{\beta}}$ and $\Psi_2^{\bar{\theta}, \bar{\beta}}$ are globally defined and the exponential affine formula (5.6) holds.

The following lemma will help us in the discussion to follow.

Lemma 5.10. *The sign of the risk premium $R_Q^F(t, \tau)$ is the same as the sign of the function*

$$\Sigma(t, \tau) \triangleq \Psi_0^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_0^{0,0}(\tau) + (\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_1^{0,0}(\tau))\sigma^2(t) + (\Psi_2^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_2^{0,0}(\tau))X(t).$$

Moreover,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \Sigma(t, \tau) &= \frac{\theta_1}{\alpha(1 - \beta_1)} + \int_0^\infty \int_0^1 \kappa'_L \left(\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2 \right) d\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) ds \\ &\quad - \int_0^\infty \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds, \end{aligned} \quad (5.18)$$

and

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} \Sigma(t, \tau) = \theta_1 + \alpha \beta_1 X(t). \quad (5.19)$$

Proof. That the sign of $R_Q^F(t, \tau)$ is the same as the sign of $\Sigma(t, \tau)$ is obvious from equation (5.5) in Theorem 5.3. From the expression for $F_P(t, T)$ in Proposition 2, we can deduce that

$$\begin{aligned} \Psi_0^{0,0}(\tau) &= \int_0^\tau \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) ds \\ &= \int_0^\tau \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds, \\ \Psi_1^{0,0}(\tau) &= e^{-\rho\tau} \frac{1 - e^{-(2\alpha - \rho)\tau}}{2(2\alpha - \rho)}, \\ \Psi_2^{0,0}(\tau) &= e^{-\alpha\tau}. \end{aligned}$$

Furthermore, by Theorem 5.7, one has that

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \Psi_0^{\bar{\theta}, \bar{\beta}}(\tau) &= \frac{\theta_1}{\alpha(1 - \beta_1)} + \int_0^\infty \kappa_L(\Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2) ds \\ &= \frac{\theta_1}{\alpha(1 - \beta_1)} + \int_0^\infty \int_0^1 \kappa'_L(\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2) d\lambda ds, \\ \lim_{\tau \rightarrow \infty} \Psi_1^{\bar{\theta}, \bar{\beta}}(\tau) &= \lim_{\tau \rightarrow \infty} \Psi_2^{\bar{\theta}, \bar{\beta}}(\tau) = 0, \end{aligned}$$

which yields equation (5.18). On the other hand, as $\Psi_2^{\bar{\theta}, \bar{\beta}}(\tau) \rightarrow 1$ and $\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau) \rightarrow 0$ when τ tends to zero, we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{d}{d\tau} (\Psi_0^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_0^{0,0}(\tau)) &= \lim_{\tau \rightarrow 0} \{ \Lambda_0^{\bar{\theta}, \bar{\beta}}(\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau), \Psi_2^{\bar{\theta}, \bar{\beta}}(\tau)) - \Lambda_0^{0,0}(\Psi_1^{0,0}(\tau), \Psi_2^{0,0}(\tau)) \} \\ &= \Lambda_0^{\bar{\theta}, \bar{\beta}}(0, 1) - \Lambda_0^{0,0}(0, 1) = \theta_1, \\ \lim_{\tau \rightarrow 0} \frac{d}{d\tau} (\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_1^{0,0}(\tau)) &= \lim_{\tau \rightarrow 0} \{ \Lambda_1^{\bar{\theta}, \bar{\beta}}(\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau), \Psi_2^{\bar{\theta}, \bar{\beta}}(\tau)) - \Lambda_1^{0,0}(\Psi_1^{0,0}(\tau), \Psi_2^{0,0}(\tau)) \} \\ &= \Lambda_1^{\bar{\theta}, \bar{\beta}}(0, 1) - \Lambda_1^{0,0}(0, 1) = 1/2 - 1/2 = 0, \\ \lim_{\tau \rightarrow 0} \frac{d}{d\tau} (\Psi_2^{\bar{\theta}, \bar{\beta}}(\tau) - \Psi_2^{0,0}(\tau)) &= \lim_{\tau \rightarrow 0} \{ \Lambda_2^{\bar{\theta}, \bar{\beta}}(\Psi_1^{\bar{\theta}, \bar{\beta}}(\tau), \Psi_2^{\bar{\theta}, \bar{\beta}}(\tau)) - \Lambda_2^{0,0}(\Psi_1^{0,0}(\tau), \Psi_2^{0,0}(\tau)) \} \\ &= \Lambda_2^{\bar{\theta}, \bar{\beta}}(0, 1) - \Lambda_2^{0,0}(0, 1) = -\alpha(1 - \beta_1) + \alpha = \alpha\beta_1, \end{aligned}$$

which yields equation (5.19). The proof is complete. \square

We now continue with investigating in more detail the different cases of our measure change.

- **Changing the level of mean reversion (Esscher transform):** Setting $\bar{\beta} = (0, 0)$, the probability measure Q only changes the levels of mean reversion for the factor X and the volatility process

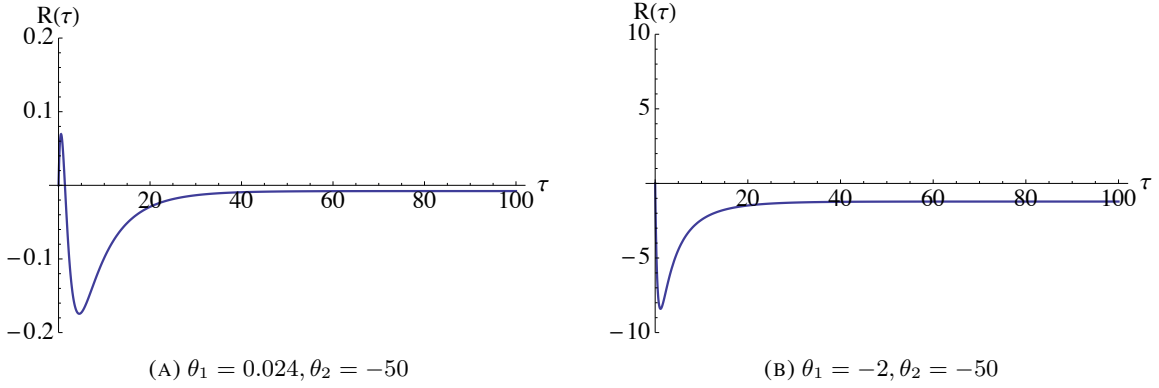


FIGURE 4. Risk premium profiles when L is a Compound Poisson process with exponentially distributed jumps. We take $\rho = 1.11, \alpha = 0.127, \lambda = 2, \varpi = 0.4, X(t) = 2.5, \sigma(t) = 0.25$ Esscher case $\beta_1 = \beta_2 = 0$.

σ^2 . Although the risk premium is stochastic, its sign is deterministic. According to Proposition 3, we have that the sign of $R_Q^F(t, \tau)$ is equal to the sign of

$$\begin{aligned} \Sigma(t, \tau) &= \theta_1 \frac{1 - e^{-\alpha\tau}}{\alpha} + \int_0^\tau \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} + \theta_2 \right) - \kappa_L(\theta_2) ds \\ &\quad - \int_0^\tau \kappa_L \left(e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} \right) ds \\ &= \theta_1 \frac{1 - e^{-\alpha\tau}}{\alpha} \\ &\quad + \theta_2 \int_0^\tau \int_0^1 \int_0^1 \kappa_L''(\lambda_2 \theta_2 + \lambda_1 e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)}) d\lambda_2 d\lambda_1 e^{-\rho s} \frac{1 - e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)} ds. \end{aligned}$$

Moreover, by Lemma 5.10, equations (5.18)-(5.19), and the fact that $\Psi_1^{\bar{\theta},0}(\tau) = e^{-\rho\tau} \frac{1 - e^{-(2\alpha-\rho)\tau}}{2(2\alpha-\rho)}$ we get

$$\lim_{\tau \rightarrow \infty} \Sigma(t, \tau) = \frac{\theta_1}{\alpha} + \theta_2 \int_0^\infty \int_0^1 \int_0^1 \kappa_L''(\lambda_2 \theta_2 + \lambda_1 \Psi_1^{\bar{\theta},0}(s)) d\lambda_2 d\lambda_1 \Psi_1^{\bar{\theta},0}(s) ds,$$

and

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} \Sigma(t, \tau) = \theta_1.$$

Note that we can write

$$\begin{aligned} &\theta_2 \int_0^\infty \int_0^1 \int_0^1 \kappa_L''(\lambda_2 \theta_2 + \lambda_1 \Psi_1^{\bar{\theta},0}(s)) d\lambda_2 d\lambda_1 \Psi_1^{\bar{\theta},0}(s) ds \\ &= \int_0^\infty \int_0^1 \int_0^1 \theta_2 \Psi_1^{\bar{\theta},0}(s) \int_0^\infty z^2 e^{(\lambda_2 \theta_2 + \lambda_1 \Psi_1^{\bar{\theta},0}(s))z} \ell(dz) d\lambda_2 d\lambda_1 ds \\ &= \int_0^\infty \int_0^\infty (e^{\theta_2 z} - 1) (e^{\Psi_1^{\bar{\theta},0}(s)z} - 1) \ell(dz) ds, \end{aligned}$$

and that $e^{\Psi_1^{\bar{\theta},0}(s)z} - 1 > 0$ for $s, z > 0$, $\Psi_1^{\bar{\theta},0}(s)$ is strictly positive. Hence, if we choose $0 < \theta_1$ small enough and $\theta_2 < 0$ large enough, we obtain a risk premium which is positive in the short end of the forward curve, and negative in the long end. Note that θ_2 must be chosen negative. Figure 4 shows graphically two possible risk premium curves for given parameters as an illustration. We recall from Benth and Ortiz-Latorre [9] that for a two-factor mean reverting

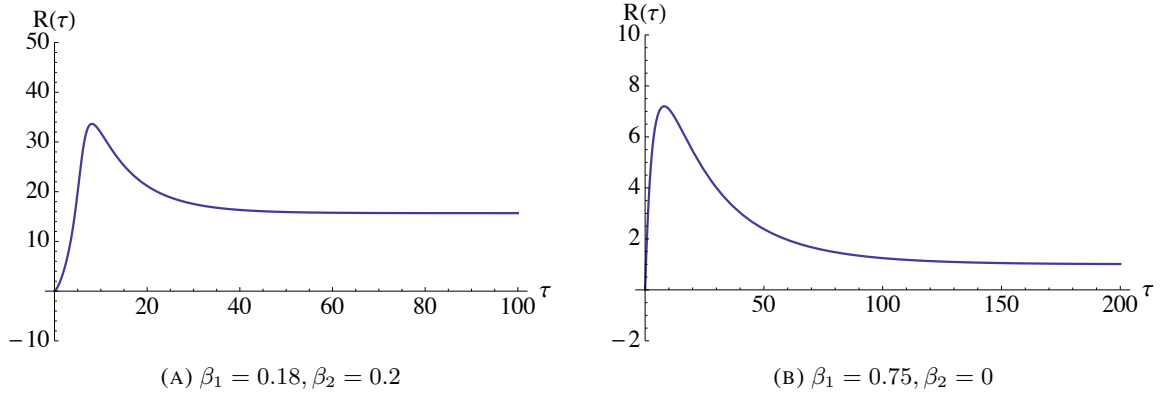


FIGURE 5. Risk premium profiles when L is a Compound Poisson process with exponentially distributed jumps. We take $\rho = 1.11, \alpha = 0.127, \lambda = 2, \bar{\sigma} = 0.4, X(t) = 2.5, \sigma(t) = 0.25$. Case $\theta_1 = \theta_2 = 0$.

stochastic dynamics of the spot price without stochastic volatility, we obtain similar deterministic risk premium profiles.

- **Changing the speed of mean reversion:** Setting $\bar{\theta} = (0, 0)$, the probability measure Q only changes the levels of mean reversion for the factor X and the volatility process σ^2 . Both the risk premium and its sign are stochastic. According to Lemma 5.10, we have that the sign of $R_Q^F(t, \tau)$ is equal to the sign of

$$\begin{aligned} \Sigma(t, \tau) \triangleq & \Psi_0^{0, \bar{\beta}}(\tau) - \int_0^\tau \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds \\ & + (\Psi_1^{0, \bar{\beta}}(\tau) - e^{-\rho\tau} \frac{1 - e^{-(2\alpha - \rho)\tau}}{2(2\alpha - \rho)}) \sigma^2(t) + (e^{-\alpha(1 - \beta_1)\tau} - e^{-\alpha\tau}) X(\tau). \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \Sigma(t, \tau) = & \int_0^\infty \int_0^1 \kappa'_L(\lambda \Psi_1^{0, \bar{\beta}}(s)) d\lambda \Psi_1^{0, \bar{\beta}}(s) ds \\ & - \int_0^\infty \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds, \end{aligned}$$

and

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} \Sigma(t, \tau) = \alpha \beta_1 X(t).$$

Note that

$$\Lambda^{0, \bar{\beta}}(u_1, u_2) - \Lambda^{0, 0}(u_1, u_2) = \frac{\rho \beta_2}{\kappa_L''(0)} (\kappa'_L(u_1) - \kappa'_L(0)) \geq 0, \quad u_1 \geq 0,$$

and using a comparison theorem for ODEs, Theorem 6.1, page 31, in Hale [16], we get that $\Psi_1^{0, \bar{\beta}}(t) \geq e^{-\rho t} \frac{1 - e^{-(2\alpha - \rho)t}}{2(2\alpha - \rho)}, t \geq 0$. Hence, the risk premium will approach to a non negative value in the long end of the forward curve. In the short end, it can be positive or negative and stochastically varying with $X(t)$. In Figure 5 we show two different risk premium curves, where we in particular notice the different convexity behaviour in the short end. As all the risk premia curves will be positive in the long end, it is not very realistic from the practical point of view to have $\bar{\theta} = (0, 0)$.

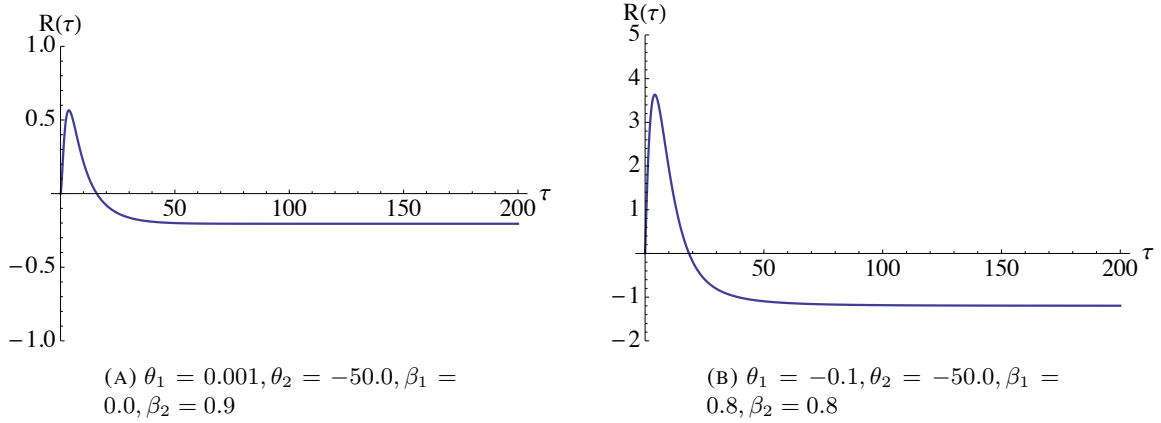


FIGURE 6. Risk premium profiles when L is a Compound Poisson process with exponentially distributed jumps. We take $\rho = 1.11, \alpha = 0.127, \lambda = 2, \frac{\rho}{\alpha} = 0.4, X(t) = 2.5, \sigma(t) = 0.25$.

- **Changing the level and speed of mean reversion simultaneously:** In the general case we modify the speed and level of mean reversion for the factor X and the volatility process σ^2 simultaneously. According to Lemma 5.10, we have that

$$\begin{aligned} \lim_{\tau \rightarrow \infty} \Sigma(t, \tau) &= \frac{\theta_1}{\alpha(1 - \beta_1)} + \int_0^\infty \int_0^1 \kappa'_L \left(\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2 \right) d\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) ds \\ &\quad - \int_0^\infty \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds, \end{aligned}$$

and

$$\lim_{\tau \rightarrow 0} \frac{d}{d\tau} \Sigma(t, \tau) = \theta_1 + \alpha \beta_1 X(t).$$

If we choose $\beta_1 = 0$, then we need to prove that for some $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ and $0 < \theta_1$ we have

$$\begin{aligned} &\int_0^\infty \int_0^1 \kappa'_L \left(\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} \right) d\lambda e^{-\rho s} \frac{1 - e^{-(2\alpha - \rho)s}}{2(2\alpha - \rho)} ds \\ &> \frac{\theta_1}{\alpha} + \int_0^\infty \int_0^1 \kappa'_L \left(\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) + \theta_2 \right) d\lambda \Psi_1^{\bar{\theta}, \bar{\beta}}(s) ds, \end{aligned} \tag{5.20}$$

in order to ensure a risk premium that changes sign from positive to negative. In fact, inequality 5.20 holds by choosing θ_1 small enough and θ_2 a large negative number, because $\lim_{\theta_2 \rightarrow -\infty} \kappa'_L(\theta_2) = 0$. See Figure 6 for two cases.

Remark 5.11. In contrast to the arithmetic case, one can get a positive risk premium for short time to maturity that rapidly changes to negative by just changing the parameters of the Esscher transform, see Figure 4. Similarly to the arithmetic case, it is not possible to get the sign change by just modifying the speed of mean reversion of the factors, see Figure 5.

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